Multi_Resolution Expansion

- Scaling fn ϕ : creates a series of approximations of a fn each differing by a factor of 2 in resolution

- Function ψ : (wavelet) encodes diff between adjacent approximations.

Series Expansion

Expand fn $f(x)$ as:

$$f(x) = \sum_{k} \alpha_k \, \phi_k(x)$$

$\phi_k(x) \triangleq$ real valued expansion functions

$\alpha_k \triangleq$ " " " coefficients

If expansion unique i.e. only one set of α_k for $f(x)$

$\Longrightarrow$ ϕ_k = basis functions. $\{\phi_k\}$ = basis for class of fns.

function space: $V \triangleq \overline{\text{Span}_k \{\phi_k(x)\}}$ closed span of expansion set

$f(x) \in V \Longrightarrow f(x)$ is in closed span of $\{\phi_k(x)\}$

and can be written as $f(x) = \sum_k \alpha_k \phi_k(x)$

Dual function $\{\tilde{\phi}_k(x)\}$ To $\{\phi_k(x)\}$

$$\alpha_k = \langle \tilde{\phi}_k(x), f(x) \rangle = \int \tilde{\phi}_k^*(x) f(x)\, dx$$

Consider 3 cases:

① Expansion fns form an orthonormal basis for V:

$$\langle \phi_j(x), \phi_k(x) \rangle = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$\Rightarrow \phi_k(x) = \check{\phi}_k(x)$ basis & dual same.

$\Rightarrow \alpha_k = \langle \phi_k(x), f(x) \rangle$

② Expansion fn orthogonal but not orthonormal

$$\langle \phi_j(x), \phi_k(x) \rangle = 0 \qquad j \neq k$$

$\Rightarrow$ basis fn and dual are bi-orthogonal

$$\alpha_k = \langle \check{\phi}_k(x), f(x) \rangle$$

$$\langle \phi_j(x), \check{\phi}_k(x) \rangle = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

③ More than one set of α_k in

$$f(x) = \sum_k \alpha_k \, \phi_k(x)$$

$\Rightarrow$ Exp fn and duals are "overcomplete"

or "redundant"

Form a frame

$$A \, \|f(x)\|^2 \leq \sum_k |\langle \phi_k(x), f(x) \rangle|^2 \leq B \, \|f(x)\|^2$$

for $A > 0$, $B < \infty$ $\forall f(x) \in V$

- If $A = B \longrightarrow$ Tight frame.

Daubechies 1992 $\quad f(x) = \frac{1}{A} \sum_k \langle \phi_k(x), f(x) \rangle \, \phi_k(x)$

Scaling functions

- Start with real, square integrable fn $\phi(\cancel{x})$
- Build a set $\phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k)$

$$j, k \in \mathbb{Z} \qquad \phi(x) \in L^2(\mathbb{R})$$

- Denote subspace ~~span~~:

$$V_j = \operatorname{span}_k \{\phi_{j,k}(x)\}$$

Then if $f(x) \in V_j \Rightarrow f(x) = \sum_k \alpha_k \phi_{j,k}(x)$

- Example Haar basis.

$$\phi(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{0 otherwise} \end{cases}$$

Show Fig 7.11 G+W 3E

$f(x) \in V_0 \Rightarrow f(x) \in V_1 \quad : \quad V_0 \subset V_1$

For Haar

Multiresolution Analysis (Mallat)

① Scaling fn is $\perp$ to its integer translates
(only for Haar)

②

$$\ldots\ldots \quad V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots$$

nesting of subspaces.

if $f(x) \in V_j$ then $f(2x) \in V_{j+1}$

③ Only fn common to all V_j is $f(x) \equiv 0$

$$V_{-\infty} = \{0\}$$

(4) Any fn can be represented with arbitrary precision.

$$V_{\infty} = \{ L^2(R) \}.$$

- Can write $\phi_{j,k}$ as linear Combination of $\phi_{j+1,k}$

$$\phi_{j,k}(x) = \sum_n \alpha_n \phi_{j+1,n}(x)$$

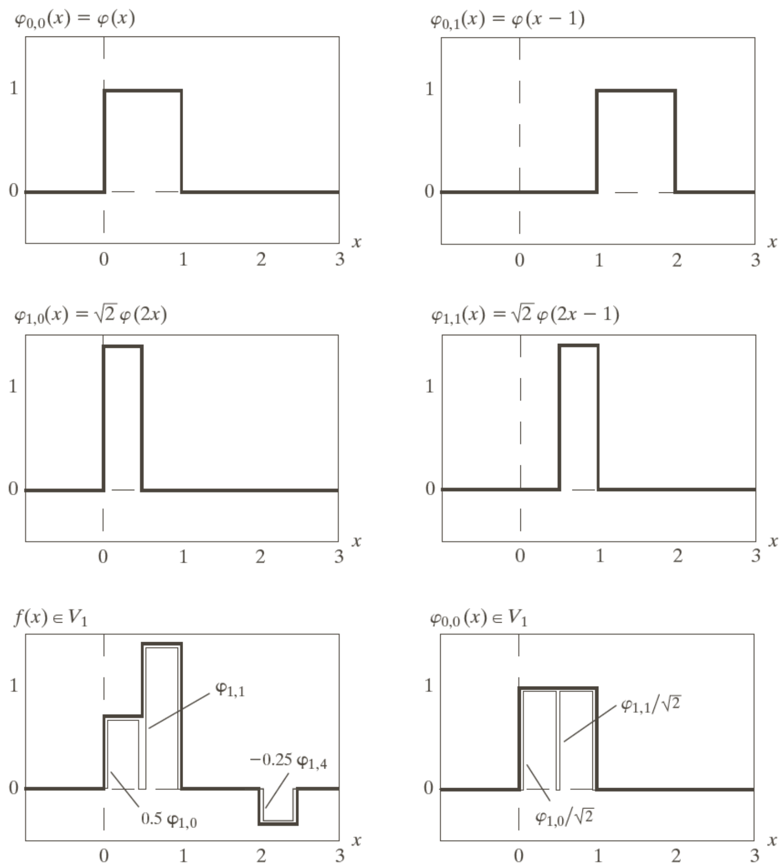
$$\phi_{j,k}(x) = \sum_n h_\phi(n)\, 2^{\frac{j+1}{2}} \phi(2^{j+1} x - n)$$

Set $j = k = 0 \Rightarrow \phi_{0,0} = \phi(x)$

$$\phi(x) = \sum_n h_\phi(n) \sqrt{2}\, \phi(2x - n)$$

- $\phi(x)$ can be built from double resolution copies of itself. ie from $\phi(2x)$
- Expansion fn of V_j is linear comb of V_{j+1}

Fig 7.11f G+W



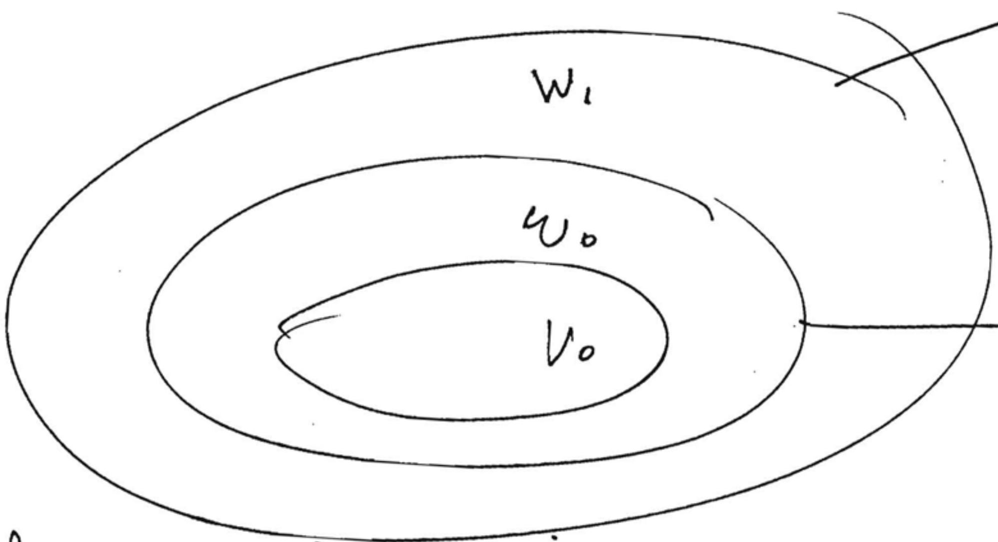
a	b
c	d
e	f

FIGURE 7.11 Some Haar scaling functions.

Wavelet fns

- Spans the difference between 2 adjacent subspaces V_j and V_{j+1}

$V_2 = V_1 \oplus W_1$
$= V_0 \oplus W_0 \oplus W_1$



$V_1 = V_0 \oplus W_0$

- $\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k) \qquad k \in Z$
- $W_j = \underset{k}{\text{Span}}\{\psi_{j,k}(x)\}$

$$V_{j+1} = V_j \oplus W_j$$

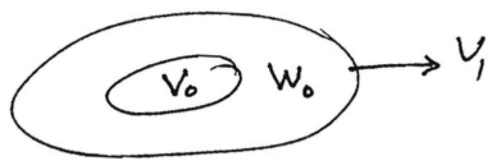
union of spaces

– Orthogonal complement of V_j in V_{j+1} is W_j

$$\Longrightarrow \langle \phi_{j,k}, \psi_{j,l} \rangle = 0 \qquad \forall j,k,l \in \mathbb{Z}$$

– $$L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots$$
$$= V_1 \oplus W_1 \oplus + W_2 \oplus \cdots$$
$$= \cdots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

no need to deal with ϕ only ψ.

– 

If $f \in V_0$

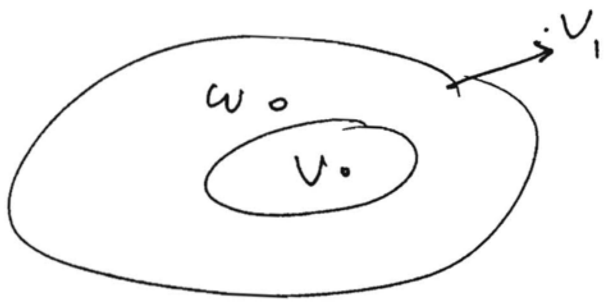
f is linear comb of scaling fn in V_0
+ linear comb of wavelet from W_0

- $L^2(R) = V_j \oplus W_j \oplus W_{j+1} \oplus \cdots\cdots$

j arbitrary.

$= V_0 \oplus W_0 \oplus W_1 \oplus W_2 + \cdots$

$= V_5 \oplus W_6 \oplus W_7 \oplus W_8 \oplus \cdots$



Basis for $V_0 \rightarrow \phi(x)$

" " $V_1 \rightarrow \phi(2x)$

" " $W_0 \rightarrow \psi(x)$

$\phi(x) \simeq$ linear comb of $\phi(2x)$

$\psi(x) \simeq$ " " " $\phi(2x)$

$$\psi(x) = \sum_n h_\psi(n) \sqrt{2}\, \phi(2x-n)$$

- Theorem by Burrus $\quad h_\psi(n) = (-1)^n h_\phi(1-n)$

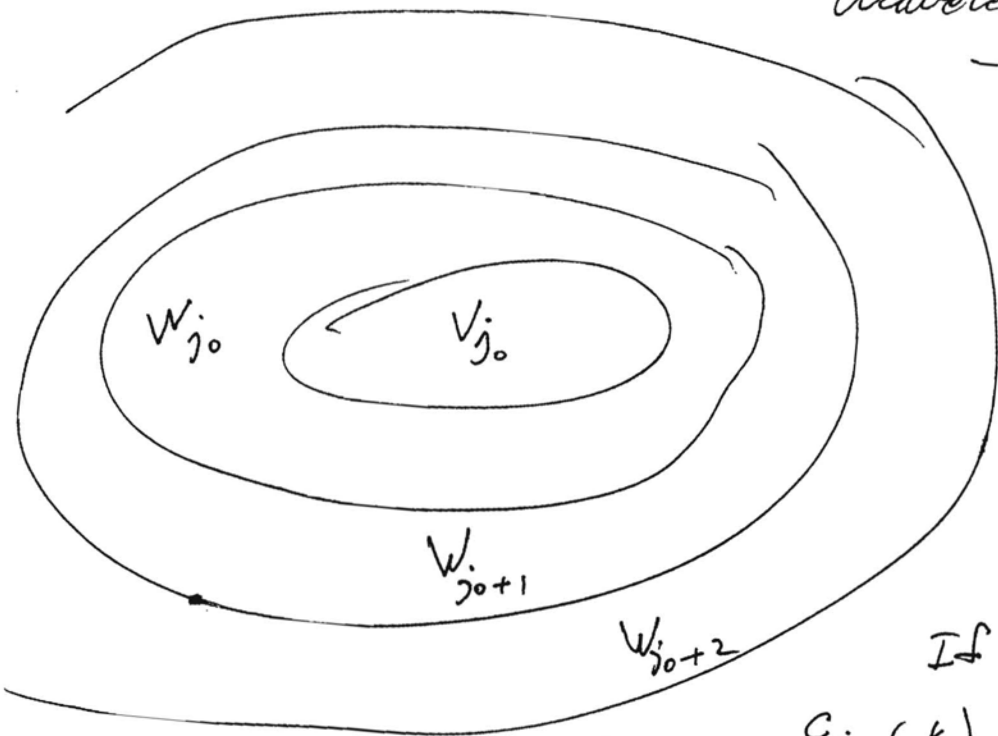
$\Longrightarrow$ from $\phi(x) \rightarrow h_\phi \rightarrow h_\psi \rightarrow \psi$

$\Rightarrow$ For Haar

$$\psi(x) = \begin{cases} 1 & 0 \leq x < 0.5 \\ -1 & 0.5 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Show Fig 7.14 G+W

Wavelet Series Expansion



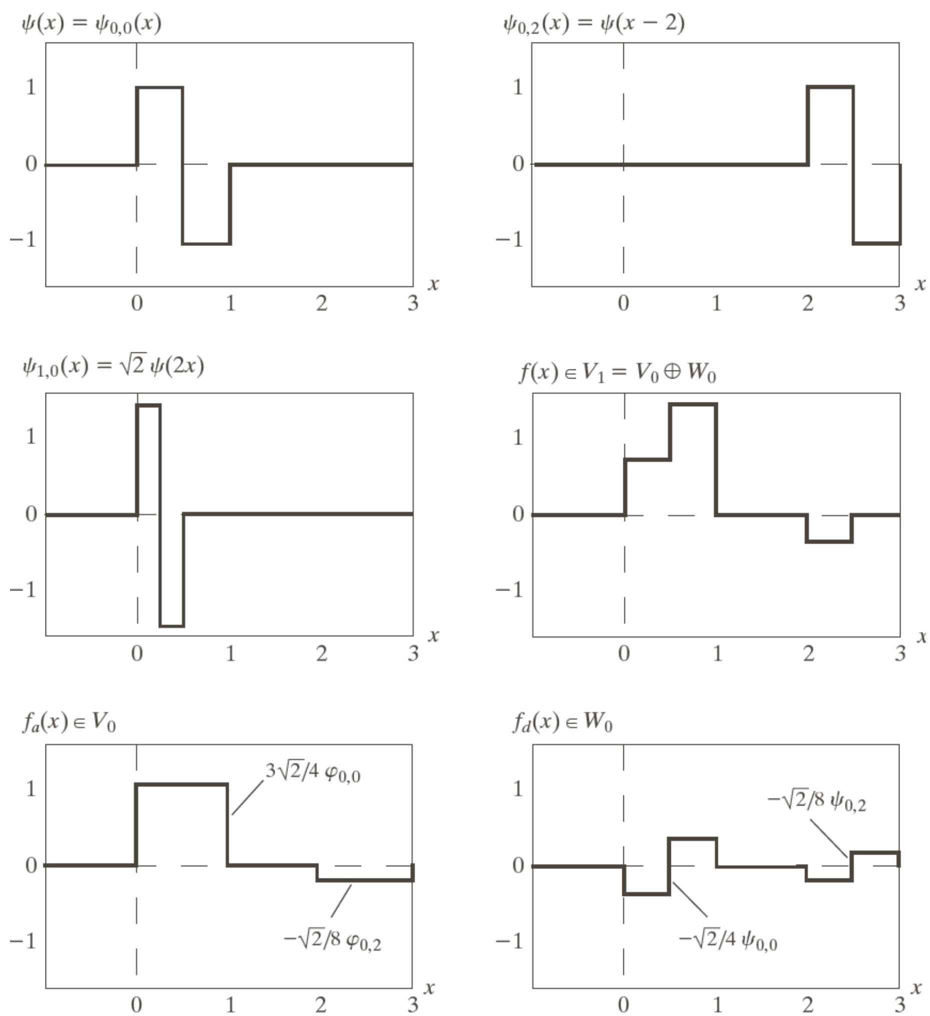
Arbitrary j_0.

$$f(x) = \sum_k c_{j_0}(k)\, \phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k)\, \psi_{j,k}(x)$$

If ϕ orthonormal or tight frame:

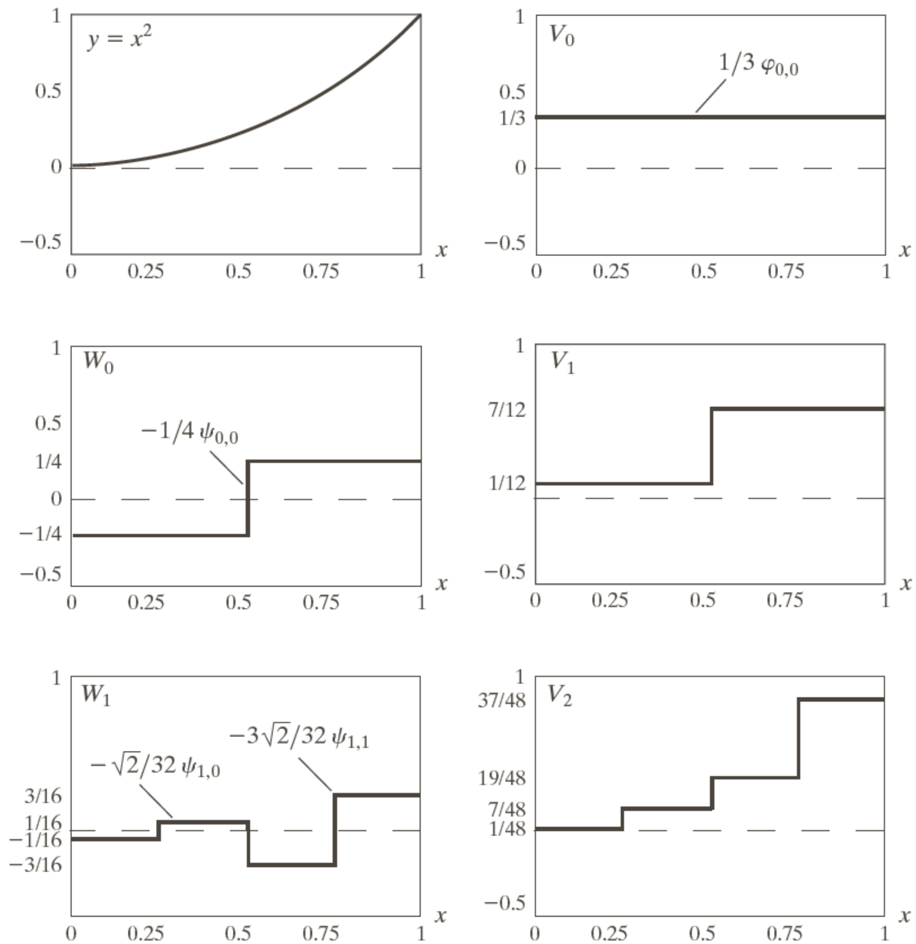
$$c_{j_0}(k) = \langle f(x), \phi_{j_0,k}(x) \rangle = \int f(x)\, \phi_{j_0,k}(x)\, dx$$

$$d_j(k) = \langle f(x), \psi_{j,k}(x) \rangle = \int f(x)\, \psi_{j,k}(x)\, dx$$



a b
c d
e f

FIGURE 7.14 Haar wavelet functions in W_0 and W_1.



a b
c d
e f

FIGURE 7.15
A wavelet series expansion of $y = x^2$ using Haar wavelets.

Show Fig 7.15

Discrete Wavelet Transform

So far ~~...~~ dealt with $f(x)$ x Real.

now deal with $f(n)$ n integer $\Longrightarrow$ sequence not fn.

_ Forward DWT coefficients for $f(n)$, (assuming Tight frame or orthogonal) sampled version of $\phi_{j_0,k}(x)$

$$W_\phi(j_0, k) = \frac{1}{\sqrt{M}} \sum_n f(n) \, \phi_{j_0,k}(n)$$

$$W_\psi(j, k) = \frac{1}{\sqrt{M}} \sum_n f(n) \, \psi_{j,k}(n) \qquad j \geq j_0$$

Then

$$f(n) = \frac{1}{\sqrt{M}} \sum_k W_\phi(j_0, k) \, \phi_{j_0,k}(n) \; +$$

$$\frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi(j, k) \, \psi_{j,k}(n)$$

Continuous Wavelet Transform

- Already discussed last Time

$$W_\psi(s, \tau) = \int_{-\infty}^{+\infty} f(x)\, \psi_{s,\tau}(x)\, dx$$

$$\psi_{s,\tau}(x) = \frac{1}{\sqrt{s}}\, \psi\left(\frac{x-\tau}{s}\right)$$

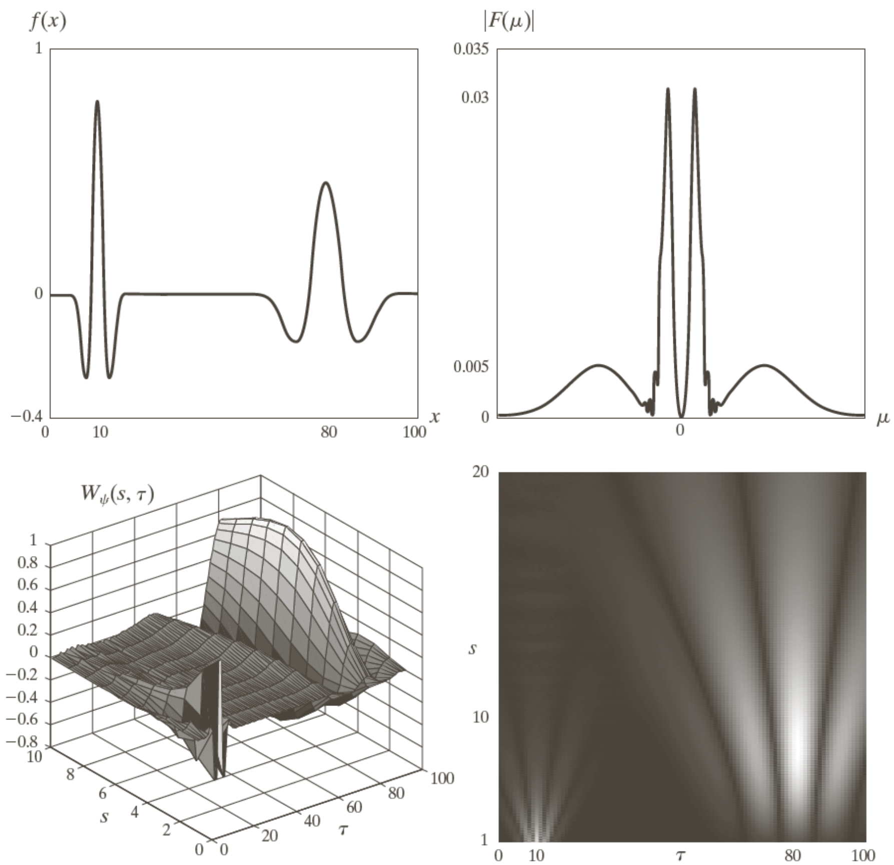
s = scale $\qquad \tau$ = Translation

Inverse Wavelet $\quad f(x) = \frac{1}{C_\psi} \int_0^{\infty} \int_{-\infty}^{+\infty} W_\psi(s,\tau)\, \frac{\psi_{s,\tau}(x)}{s^2}\, d\tau\, ds$

where $\quad C_\psi = \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\mu)|^2}{|\mu|}\, d\mu \quad \leftarrow$ admissibility criterion.

- $s \propto \frac{1}{2^j}$ $\qquad$ - compression $\quad 0 < s < 1$
  - dialation $\quad s > 1$

- Show Fig 7.16 G+W



a b
c d

FIGURE 7.16 The continuous wavelet transform (c and d) and Fourier spectrum (b) of a continuous 1-D function (a).

Fast Wavelet Transform

$$\phi(x) = \sum_n h_\phi(n)\,\sqrt{2}\,\phi(2x-n)$$

$$x \leftarrow 2^j$$

$$\phi(2^j x - k) = \sum_n h_\phi(n)\,\sqrt{2}\,\phi\big(2(2^j x - k) - n\big)$$

$$= \sum_m h_\phi(m-2k)\,\sqrt{2}\,\phi(2^{j+1} x - m)$$

Similarly

$$\left.\begin{array}{l} \psi(2^j x - k) = \sum_m h_\psi(m-2k)\,\sqrt{2}\,\phi(2^{j+1} x - m) \\ \text{Recall} \quad d_j(k) = \int f(x)\, 2^{j/2}\, \psi(2^j x - k)\, dx \end{array}\right\} \Rightarrow$$

$$d_j(k) = \sum_m h_\psi(m-2k)\, c_{j+1}(m)$$

Similarly

$$c_j(k) = \sum_m h_\phi(m-2k)\, c_{j+1}(m)$$

$$c_j(k) \longrightarrow W_\phi(j,k)$$
$$d_j(k) \longrightarrow W_\psi(j,k)$$
} DWT

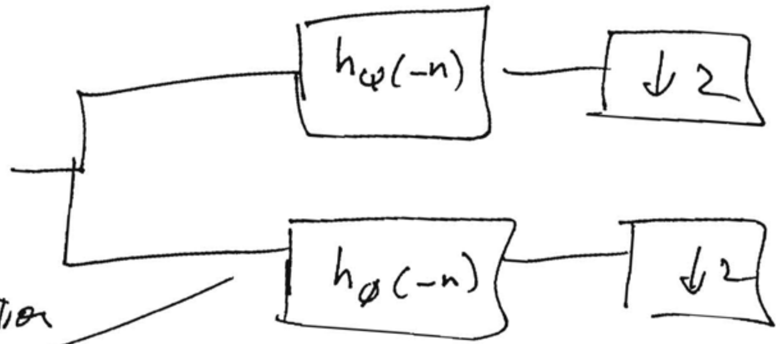
as $f(x) \longrightarrow f(n)$

Then

$$\left.\begin{aligned} W_\psi(j,k) &= \sum_m h_\psi(m-2k)\, W_\phi(j+1,m) \\ W_\phi(j,k) &= \sum_m h_\phi(m-2k)\, W_\phi(j+1,m) \end{aligned}\right\}$$

$$\Rightarrow \left\{\begin{aligned} W_\psi(j,k) &= h_\psi(-n) * W_\phi(j+1,n) \Big|_{n=2k,\ k\geq 0} \\ W_\phi(j,k) &= h_\phi(-n) * W_\phi(j+1,n) \Big|_{n=2k,\ k\geq 0} \end{aligned}\right.$$

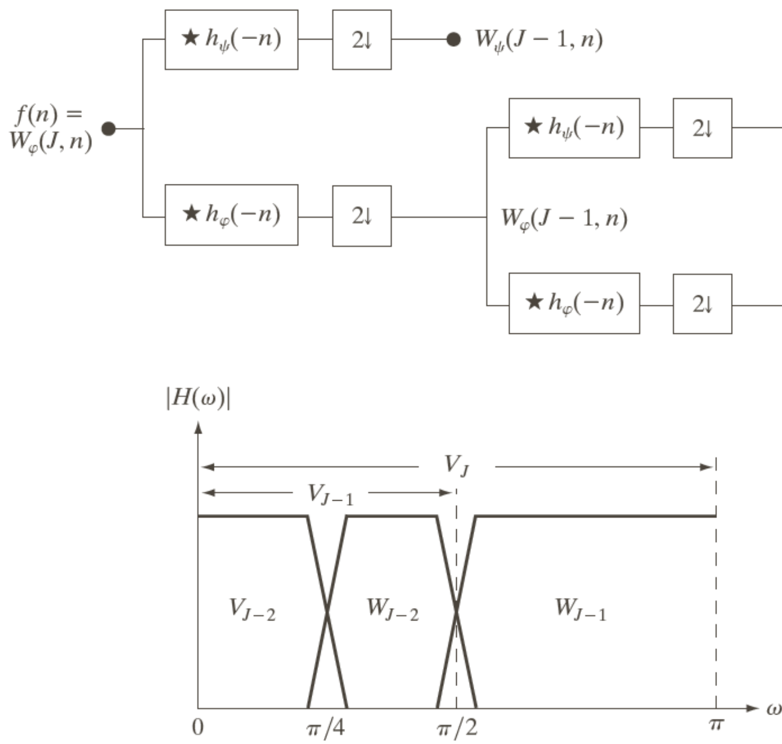
$W_\phi(j+1,n)$



Subband Analysis

Since Convolution Can use FFT or other Fast Algorithm

Show Fig 7.18



a
b

FIGURE 7.18
(a) A two-stage or two-scale FWT analysis bank and (b) its frequency splitting characteristics.