Reconstruction of 2-D Signals from Partial Fourier Information

- motivation: electron microscopy.
  optical astronomy
  - crystallography . . .

- 3 problems:
  1. FTM = magnitude of Fourier Transform
  2. Recon from Phase Fourier Transform
  3. Recon from level crossings

Recon from FTM

1_D Case:

- $x(n)$: discrete time - 1-D -
  $N+1$ nonzero points.
  $n = 0, \ldots, N$

$$e^{-j\omega} = y$$

$$\text{D.T.F.T}\{x(n)\} = X(\omega) = \sum_{n=0}^{N} x(n) e^{-j\omega n}$$

If We know $|X(\omega)|$ ⟶ Can we get $x(n)$?

continuous valued

If I know $|X(\omega)|$ at $N+1$ pts ⟹ cannot recon signal

$$
\begin{bmatrix}
e^{-0 \cdot j\omega_0} & e^{-j1\omega_0} & e^{-j2\omega_0} & \cdots \\
e^{-0 \cdot j\omega_1} & e^{-j1 \cdot \omega_1} & e^{-j2\omega_1} & \cdots e^{-j2\omega_1} \\
\vdots & & & \\
e^{-0 \cdot j\omega_N} & e^{-j\omega_N} & e^{-j2\omega_N} & \cdots e^{-jN\omega_N}
\end{bmatrix}
\begin{bmatrix}
x(0) \\ x(1) \\ \vdots \\ x(N)
\end{bmatrix}
=
\begin{bmatrix}
|X(\omega_0)|\phi \\ |X(\omega_1)|\phi \\ \vdots \\ |X(\omega_N)|\phi
\end{bmatrix}
$$

$$\sum_{n=0}^{N} x(n)\, y^n = X(\omega)$$

Sampling a 2-D poly of degree at arbitrary N points results in unique recon of polynomial

$$\underbrace{\begin{bmatrix} e^{-0jw_0} & e^{-j1w_0} & \cdots & \cdots \\ e^{-0jw_1} & \cdots & \cdots & \cdots \\ \vdots & & & \vdots \\ e^{-0jw_N} & e^{-j1w_N} & \cdots & \cdots \end{bmatrix}}_{4N} \underbrace{\begin{bmatrix} X(0) \\ \vdots \\ X(N) \end{bmatrix}}_{N} = \underbrace{\begin{bmatrix} |X(w_0)|^2 \\ |X(w_1)|^2 \\ \vdots \\ |X(w_N)|^2 \end{bmatrix}}_{4N}$$

Q: Can we uniquely reconstruct if we oversampled F.D.?

A for 1D signal $\longrightarrow$ No 2^N ambiguity

MD signals $\longrightarrow$ yes

1-D

auto-correlation fn of $x(n)$.

$$r(n) = \sum_{l} x(l)\, x^*(l+n)$$

$$R(\omega) = \text{D.T.F.T}\{r(n)\} = \sum r(n) e^{-j\omega n} = |X(\omega)|^2 =$$

$$\Longrightarrow F^{-1}\{|X(\omega)|^2\} = r(n)$$

Q: Can we obtain $x(n)$ from $r(n)$?

$$\text{Z.T.}\{r(n)\} = R(z) = X(z)\, X^*\left(\frac{1}{z^*}\right) = \sum_{n=-N}^{+N} r(n) z^{-n}$$

Associated polynomial to $R(z) \triangleq P_r(z) = \sum_{n=0}^{2N} r[N-n]\, z^n$

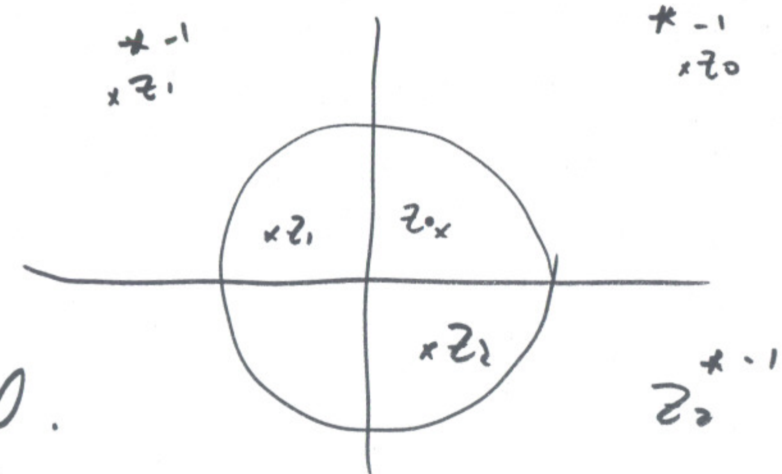
symmetric

Since coeff $P_r(z)$ are symmetric $\Longrightarrow$

If z_0 is a zero of $P_r(z)$, so is z_0^{*-1}

$\Rightarrow$ Can factor $P_r(z)$

$$P_r(z) = A \prod_{i=1}^{N} (z - z_i)(1 - z_i^* z)$$



Def mirror of a polynomial.

Associated with any polynomial, there is a mirror polynomial consisting of coefficients in reverse order and conjugated

If $P(z) = \sum_{n=0}^{N} p_n z^n \xRightarrow{\text{mirror}} \hat{p}(z) = \sum_{n=0}^{N} p_{N-n}^* z^n$

- Assume $y(n) \longrightarrow$ autocorrelation $r(n)$

$$P_r(z) = P_y(z)\, \hat{P}_y(z)$$

2^N ways in order to generate $P_y(z)$.

$$P_y(z) = \sqrt{A} \prod_{i \in I} (z - z_i) \prod_{j \notin I} (z - z z_j^*)$$

I any subset of $[1, \dots, N]$

$$\left(z_0 , z_0^{*-1} \right) \qquad P(z)$$

$$\left(z_1 , z_1^{*-1} \right) \qquad P_r(z)$$

$$\vdots$$

$$r(n)$$

$$\downarrow$$

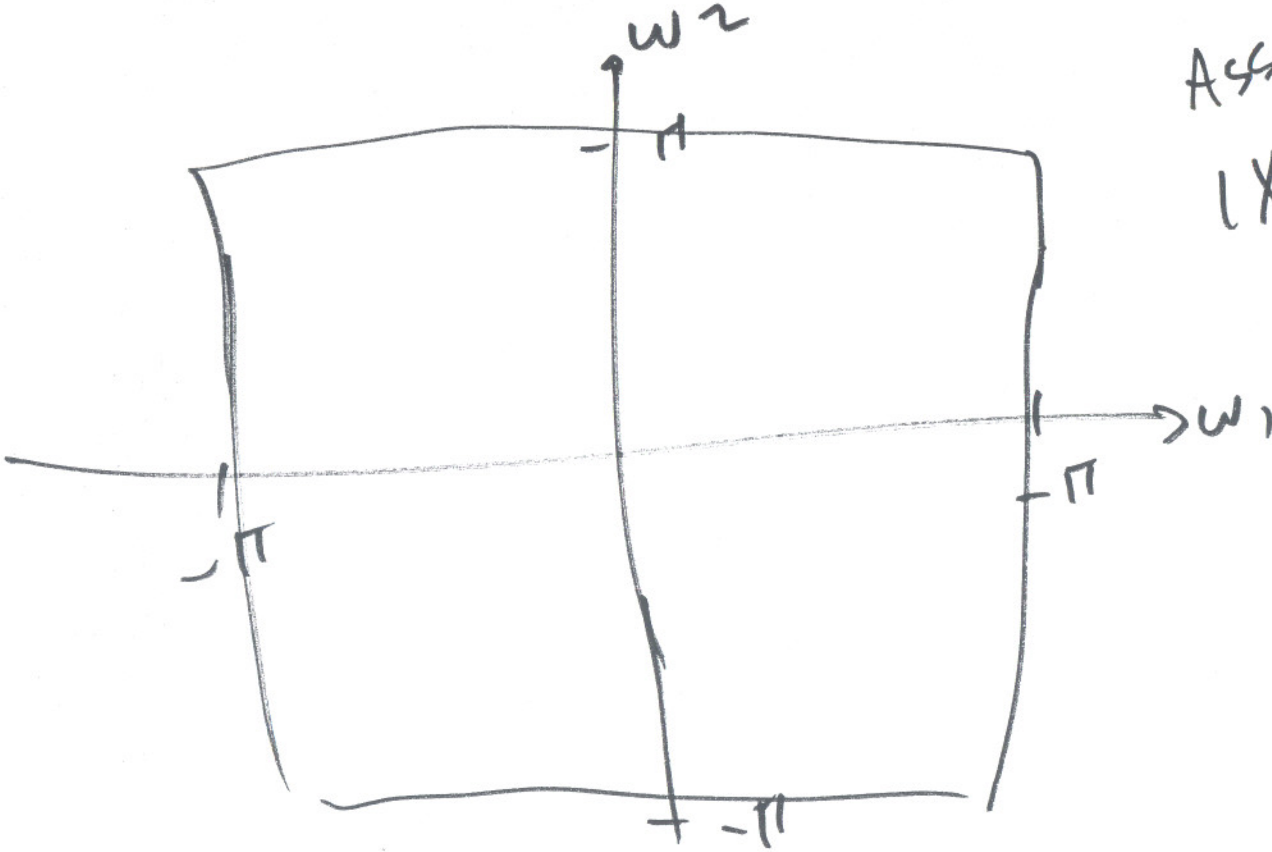
$$x(n)$$

$$\left(z_N , z_N^{*-1} \right)$$

$$2^N \quad x(n) \longrightarrow \text{same } r(n)$$

2-D case FTM

Assume $|X(\omega_1, \omega_2)|^2$



$x(n_1, n_2)$: $N \times N$

$\Rightarrow$ in 2-D, unlike 1-D, most 2D polynomials are irreducible $\Longrightarrow$ non factorable.

1982, M. Hayes : If $x(n_1, n_2)$ has irreducible associated polynomials, then all other $y(n)$ which have same F.T.M are equivalent to x. Equivalent means:

$$y(n_1, n_2) \sim x(n_1, n_2) \quad \text{if}$$

$$\cancel{y(n_1, n_2) = e^{j\theta} x(n_1, n_2)}$$

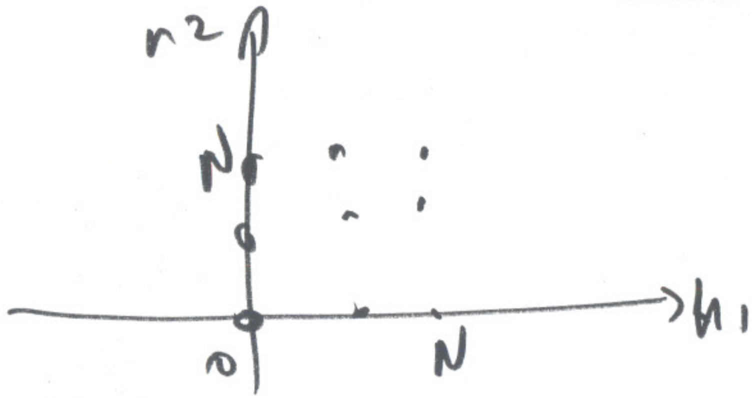
$$y(n_1, n_2) = e^{j\theta} x(k_1 \pm n_1, k_2 \pm n_2)$$

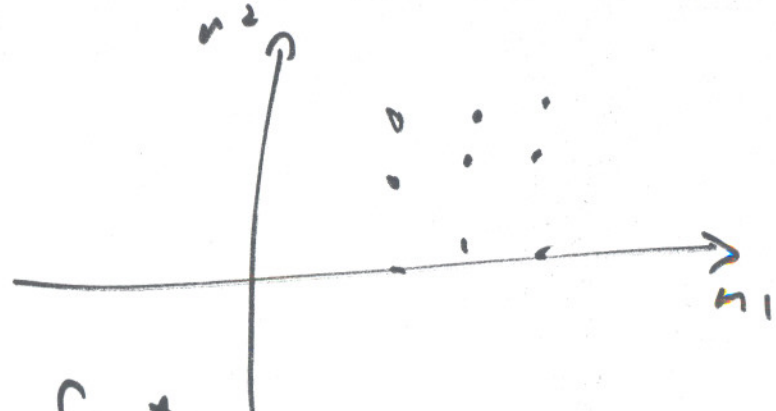
- observation (1) : $x(n_1, n_2) \longrightarrow |X(\omega_1, \omega_2)|^2$

$$e^{j\theta} x(n_1, n_2) \longrightarrow |X(\omega_1, \omega_2)|^2$$

- observation (2) $x(n_1, n_2) \longrightarrow |X(\omega_1, \omega_2)|^2$

Shift by k_1, k_2 in space domain $\Longrightarrow$ same FTM





- Observation 3: Rotatiation factor.

$x(n_1,n_2) \longrightarrow |X(\omega_1,\omega_2)|^2$

$y(n_1,n_2) = x(N-n_1, N-n_2)$

also has same FTM.

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① Assume signal $x(n_1,n_2)$ is real

$e^{j\theta}$, $\theta = 0$, or $\theta = \pi$

- Assume signal is positive $\Longrightarrow \theta = 0$

(2) - Assumption: extent of $x(n_1, n_2)$ is known.

(3) Z.T of signal x $\Longrightarrow$ irreducible.

$\xrightarrow{\text{~~Hayes~~}}$ Nail signal to factor of 2:

either $\underline{x(n_1, n_2)}$ or $x(\underline{N - n_1, N - n_2})$

same FTM.

(1) assume $\overset{x}{\vee}$ Real, ~~is~~ ~~kn~~ and positive.
Know extent.

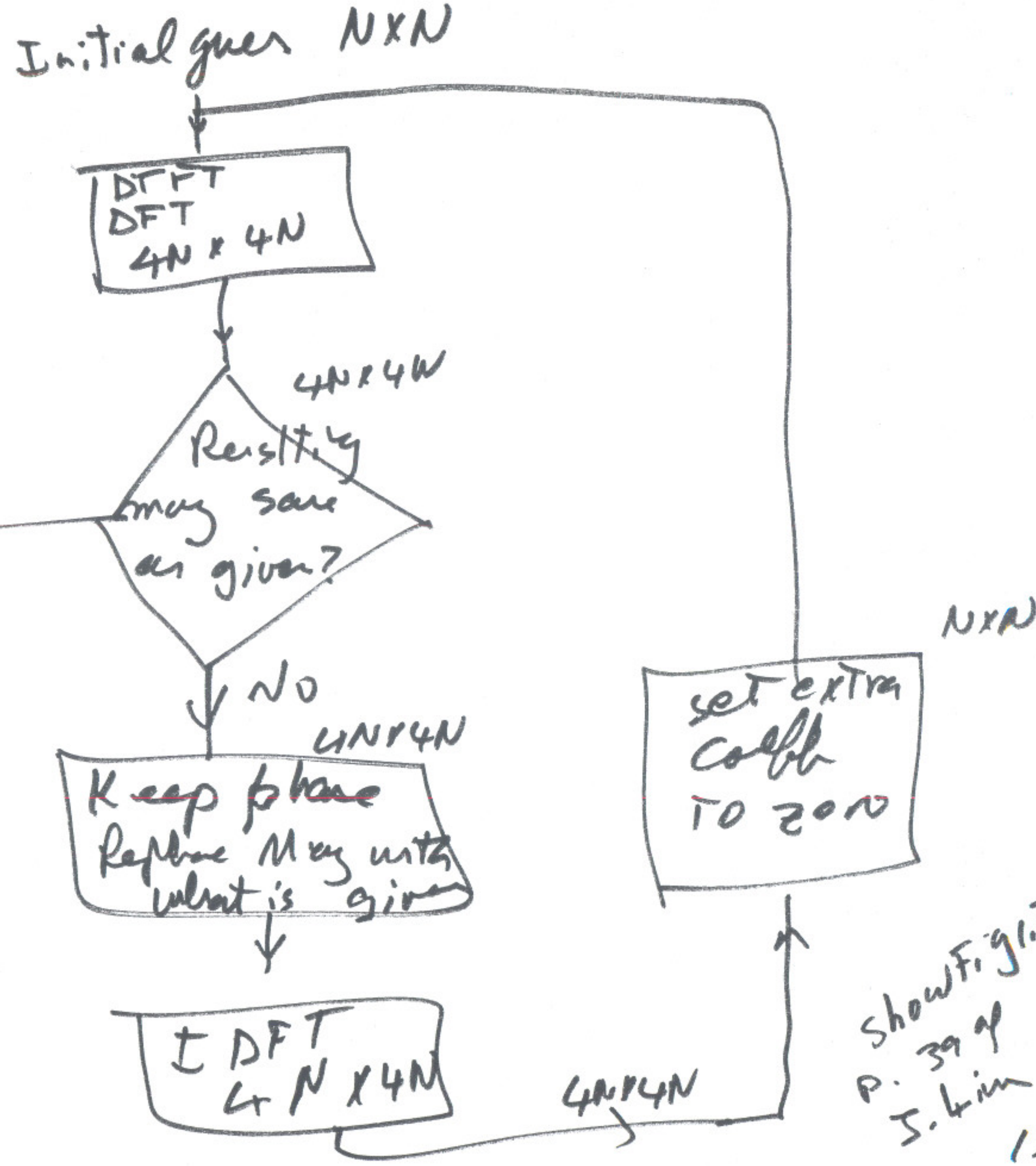
FTM can be used to recover either x or its reflection "uniquely"

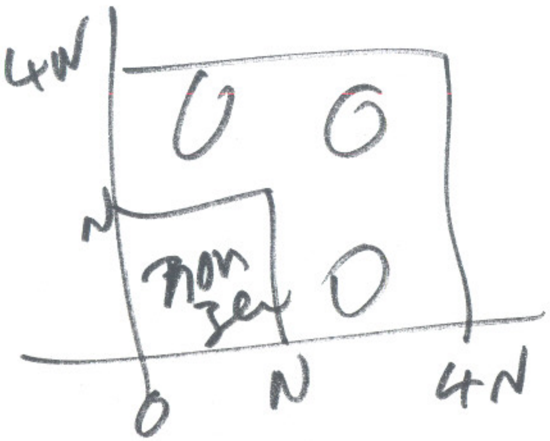
proposed iterative Alg

Assume known
4N X 4N samples
of $|X(\omega_1, \omega_2)|^2$

N X N $\quad n_1 : 0 \rightarrow N-1$
$\quad\quad\quad n_2 : 0 \rightarrow N-1$

x is positive real





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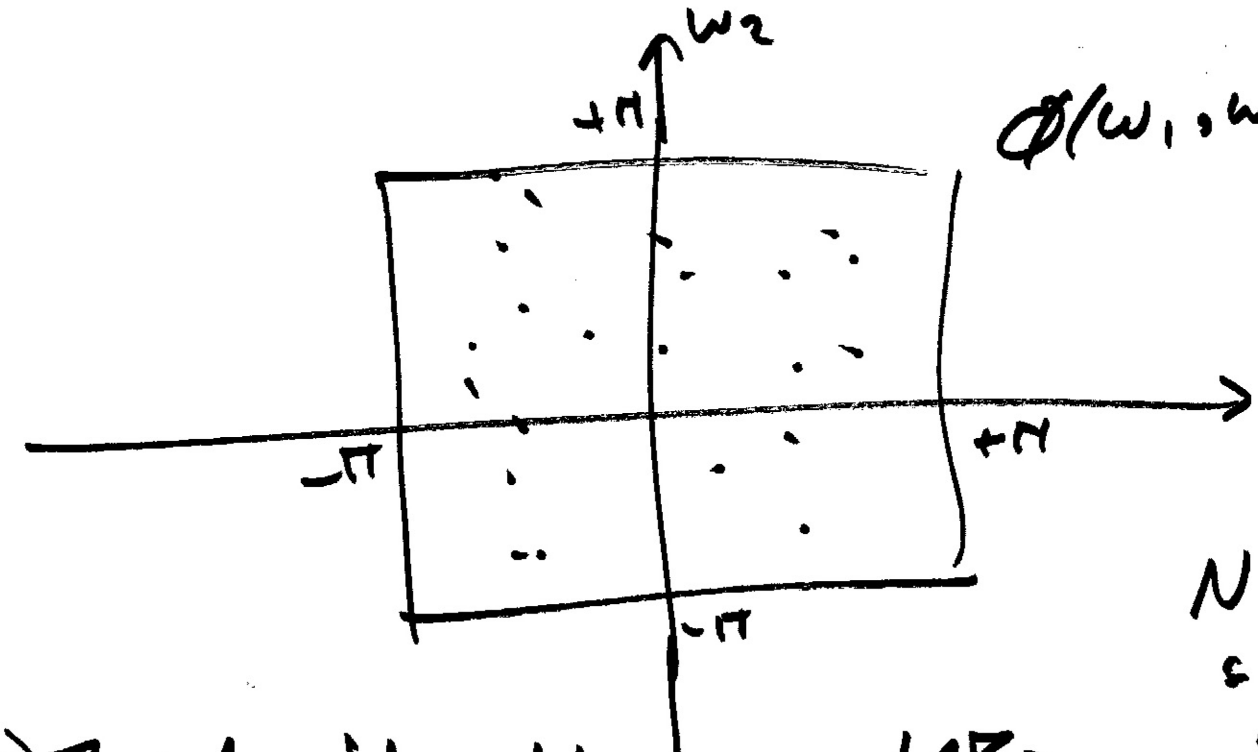
Recon from F.T. Phase

$$x(n_1,n_2) \xrightarrow{\text{D.T.F.T}} X(\omega_1,\omega_2) = |\quad| \, e^{j \measuredangle \phi}$$

$N \times N$

$$X(\omega_1,\omega_2) = \sum_{n_1} \sum_{n_2} x(n_1,n_2)\, e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$



$\phi(\omega_1,\omega_2)$

ω_1 → continuous valued variables

Need to have samples of ϕ at more than N^2

Patrick Van Hove $\approx$ 1982

Two Alg. → iterative.

→ direct.

~~the~~ 1982 → Even quantizing phase of F.T. to <u>one bit</u> & yet reconst. signal successful.

Iterative

x = original signal

- $N \times N$ signal.
- Know ROS: $N \times N$.
- → samples at $2N \times 2N$ of phase of F.T. ϕ_x