EE123
Digital Signal Processing

Lecture 5B
Time-Frequency Tiling

M. Lustig, EECS UC Berkeley

FREQUENCY ANALYSIS

- Frequency Spectrum
  - Be basically the frequency components (spectral components) of that signal
  - Show what frequencies exists in the signal
- Fourier Transform (FT)
  - One way to find the frequency content
  - Tells how much of each frequency exists in a signal

$$X(k+1) = \sum_{n=0}^{N-1} x(n+1) \cdot W_N^{kn}$$

$$x(n+1) = \frac{1}{N} \sum_{k=0}^{N-1} X(k+1) \cdot W_N^{-kn}$$

$$w_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-2j\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{2j\pi ft} df$$

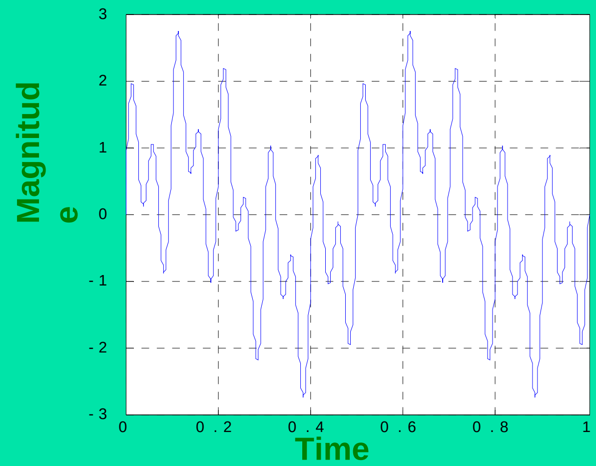
STATIONARITY OF SIGNAL

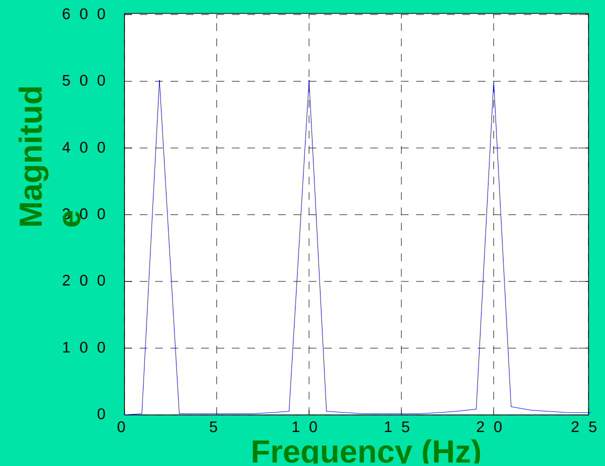
- Stationary Signal
  - Signals with frequency content unchanged in time
  - All frequency components exist at all times

- Non-stationary Signal
  - Frequency changes in time
  - One example: the "Chirp Signal"

STATIONARITY OF SIGNAL

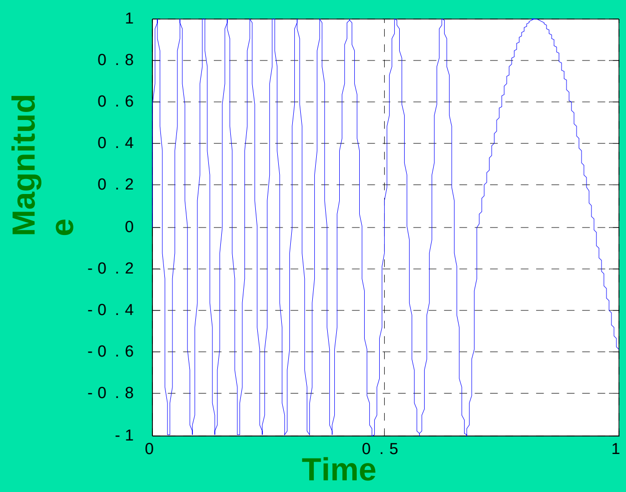
2 Hz + 10 Hz + 20Hz

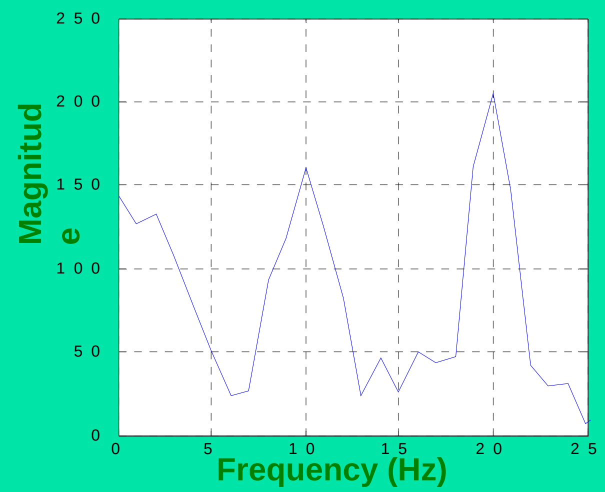




Stationary

0.0-0.4: 2 Hz +
0.4-0.7: 10 Hz +
0.7-1.0: 20Hz





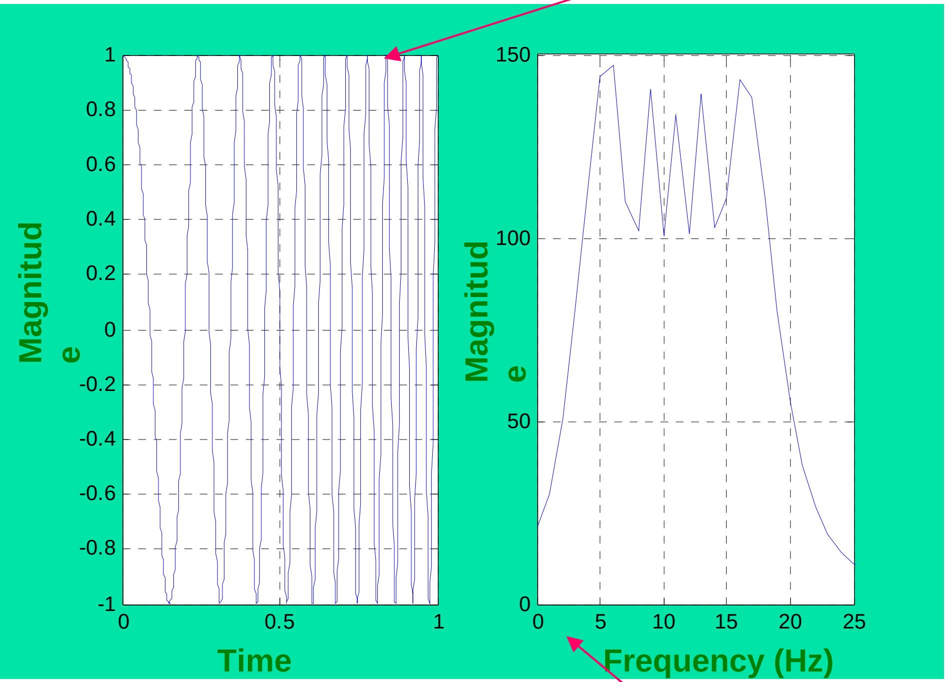
Non-Stationary

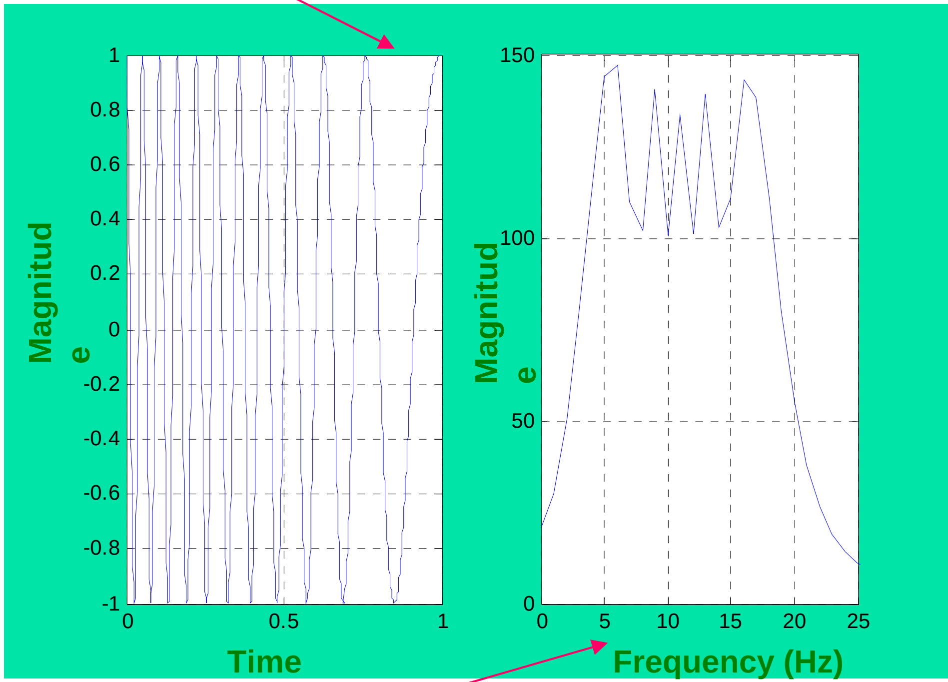
CHIRP SIGNALS

Frequency: 2 Hz to 20 Hz

Different in Time Domain

Frequency: 20 Hz to 2 Hz





Same in Frequency Domain

At what time the frequency components occur? FT can not tell!

NOTHING MORE, NOTHING LESS

- FT Only Gives what Frequency Components Exist in the Signal
- The Time and Frequency Information can not be Seen at the Same Time
- Time-frequency Representation of the Signal is Needed

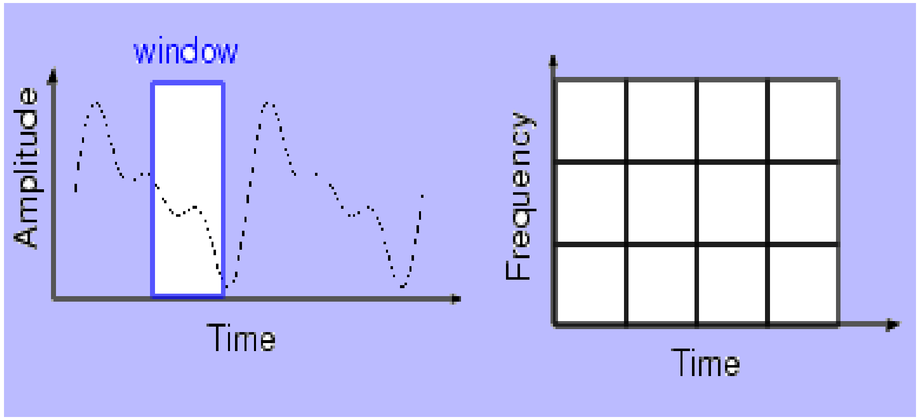
Most of Transportation Signals are Non-stationary.

(We need to know whether and also when an incident was happened.)

ONE EARLIER SOLUTION: SHORT-TIME FOURIER TRANSFORM (STFT)

SFORT TIME FOURIER TRANSFORM (STFT)

- Dennis Gabor (1946) Used STFT
  - To analyze only a small section of the signal at a time -- a technique called *Windowing the Signal*.
- The Segment of Signal is Assumed *Stationary*
- A 3D transform



$$\mathrm{STFT}_X^{(\omega)}(t', f) = \int_t \left[x(t) \bullet \omega^*(t - t') \right] \bullet e^{-j2\pi ft} \, dt$$

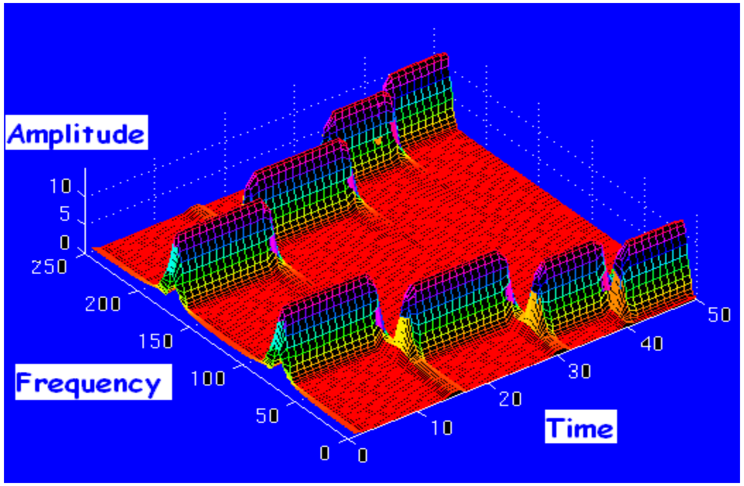
$\omega(t)$: the window function

A function of time and frequency

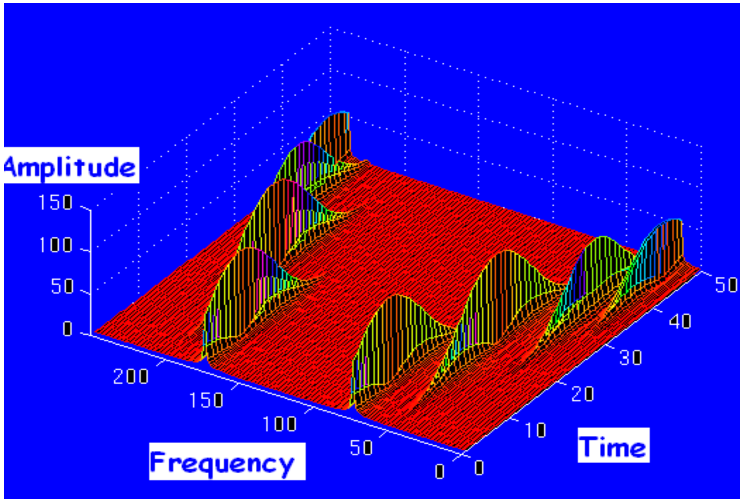
DRAWBACKS OF STFT

- Unchanged Window
- Dilemma of Resolution
  - Narrow window -> poor frequency resolution
  - Wide window -> poor time resolution
- Heisenberg Uncertainty Principle
  - Cannot know what frequency exists at what time intervals

Via Narrow Window

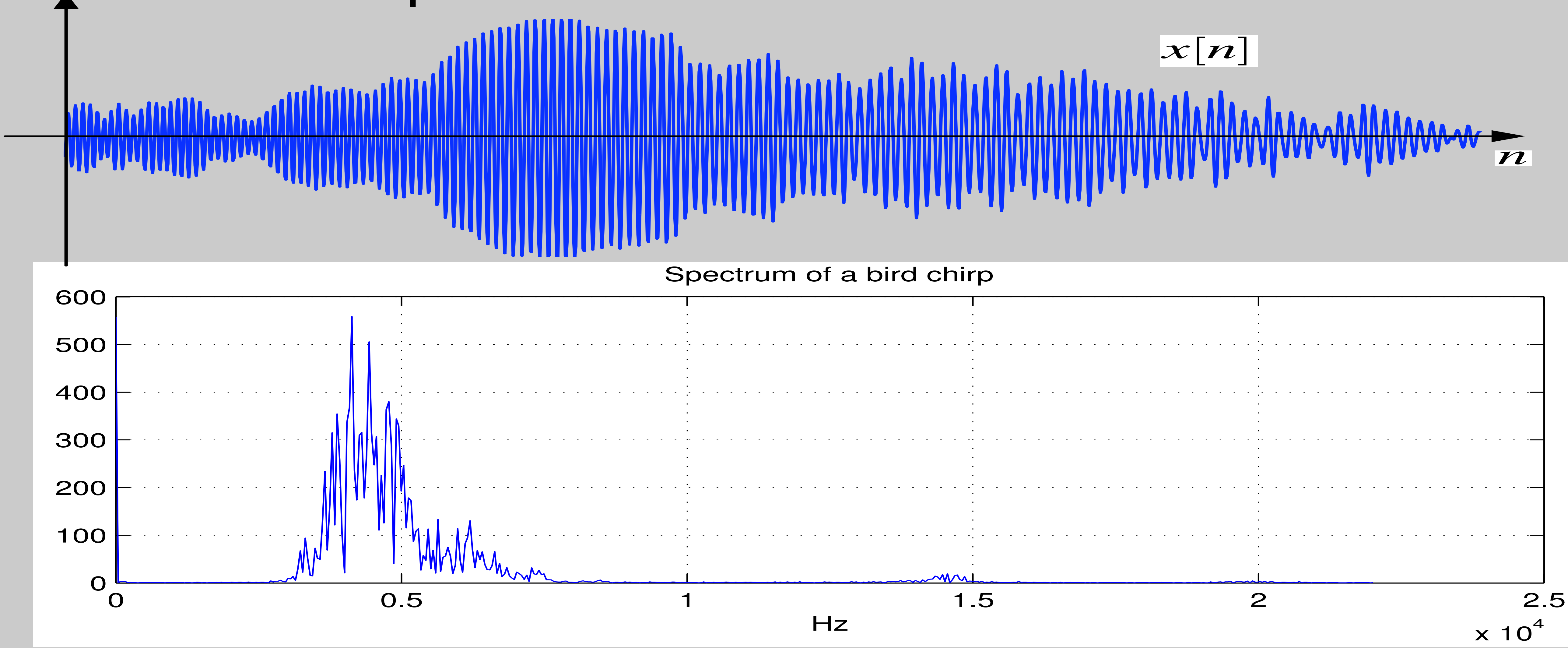


Via Wide Window



Example of spectral analysis

- Spectrum of a bird chirping
  - Interesting,.... but...
  - Does not tell the whole story
  - No temporal information!



Time Dependent Fourier Transform

- To get temporal information, use part of the signal around every time point

$$X[n,\omega) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\omega m}$$

*Also called Short-time Fourier Transform (STFT)

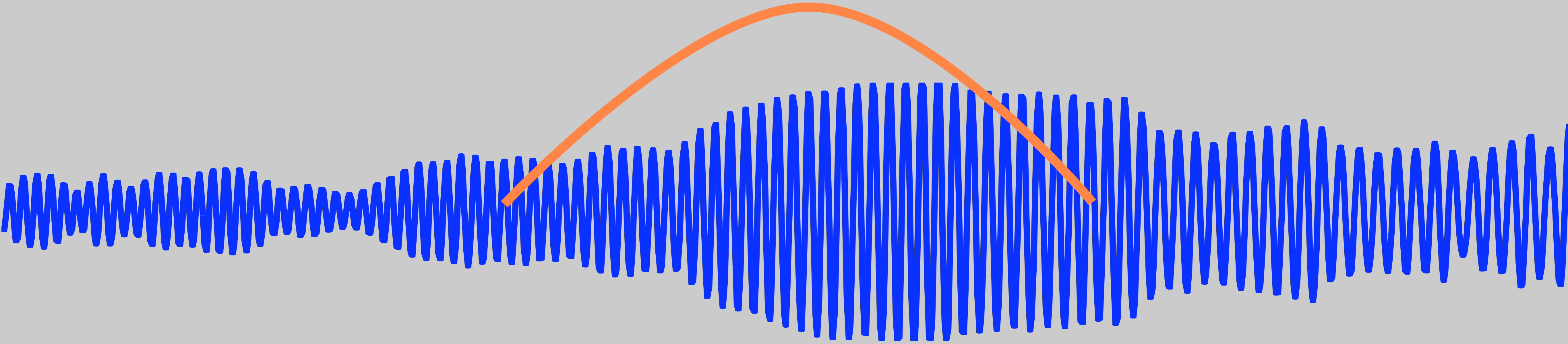
- Mapping from 1D ⇒ 2D, n discrete, *w* cont.

Time Dependent Fourier Transform

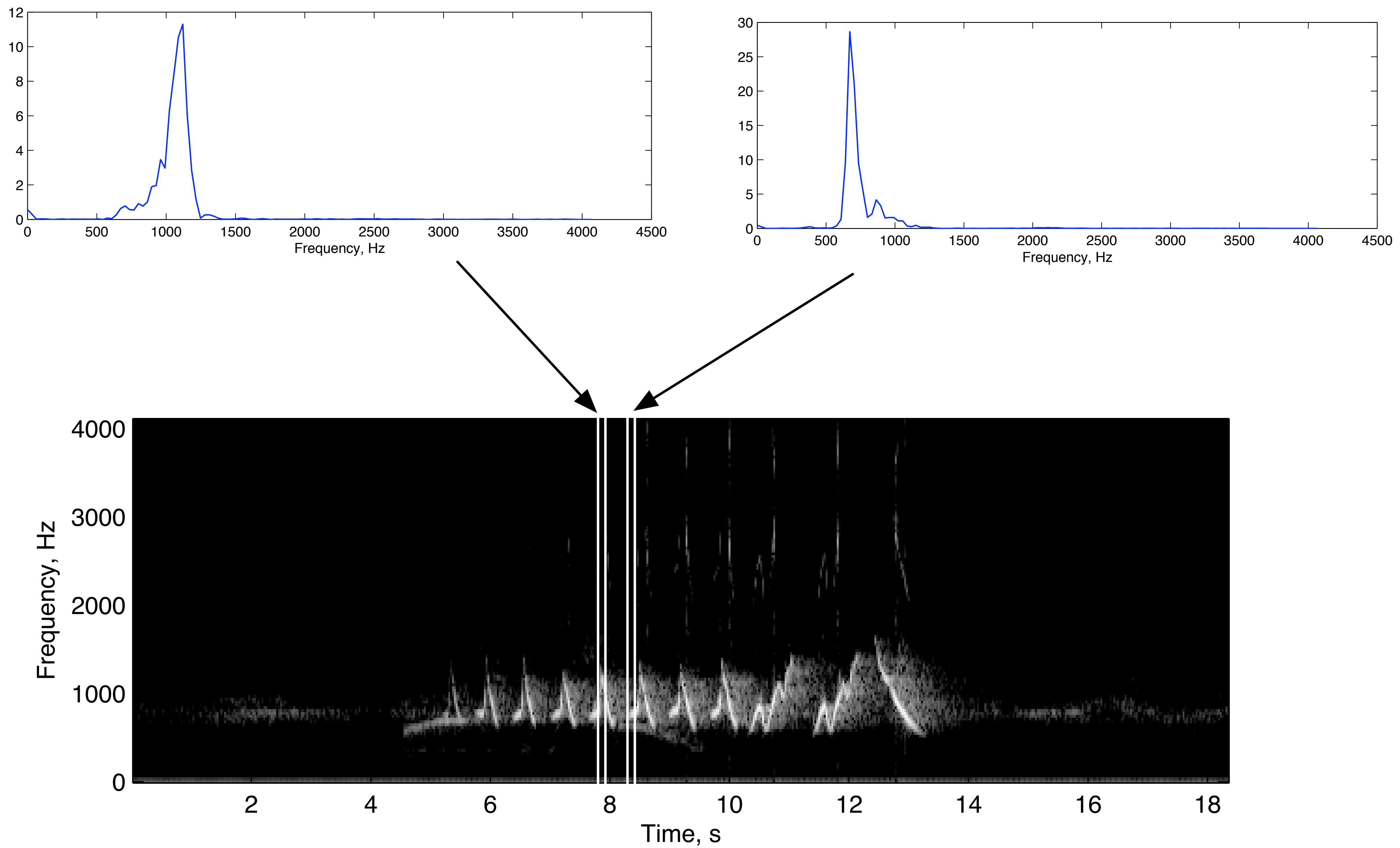
- To get temporal information, use part of the signal around every time point

$$X[n,\omega) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\omega m}$$

*Also called Short-time Fourier Transform (STFT)



Spectrogram



Discrete Time Dependent FT

$$X_r[k] = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j2\pi km/N}$$

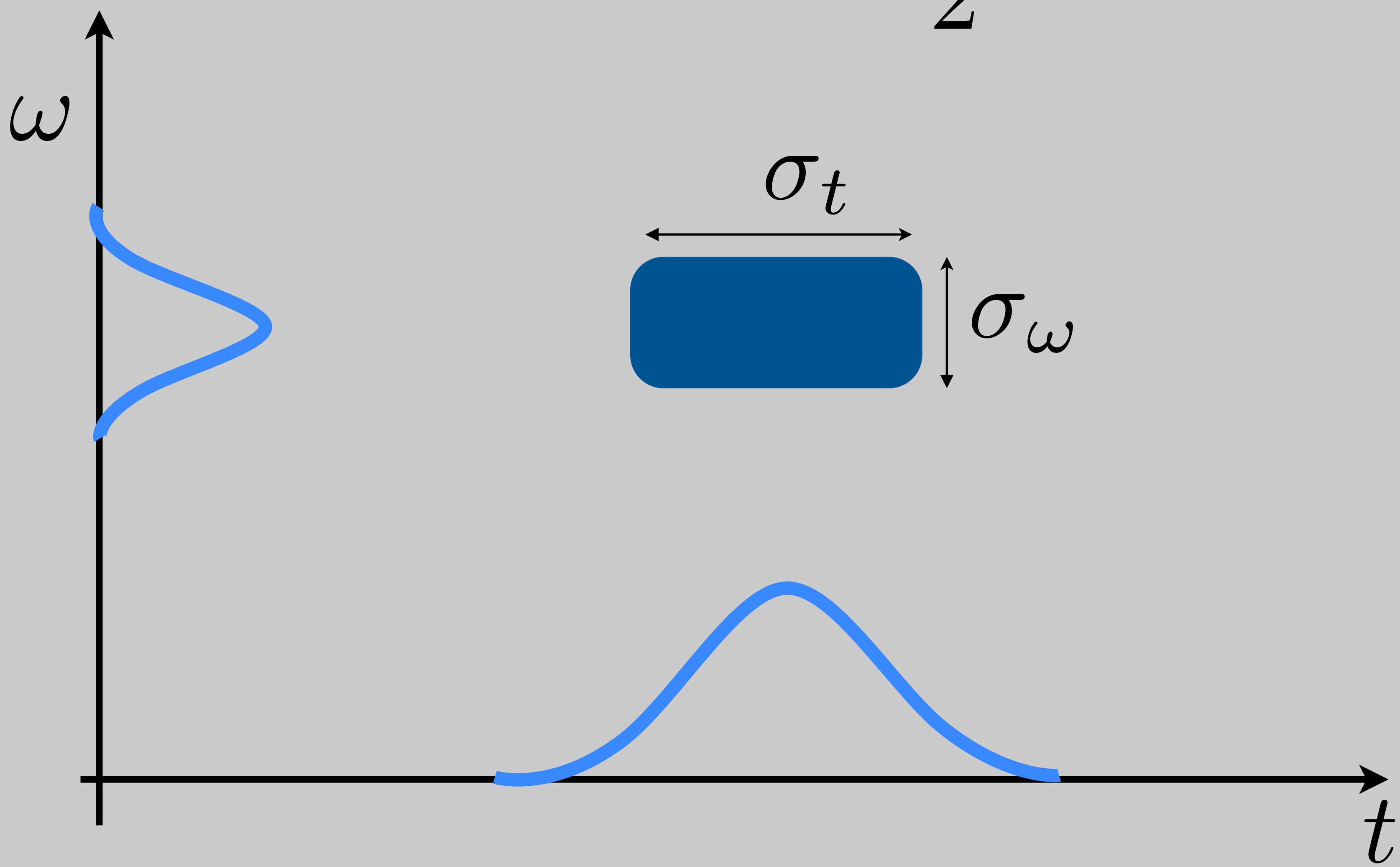
- L - Window length
- R - Jump of samples
- N - DFT length

- Tradeoff between time and frequency resolution

Heisenberg Boxes

- Time-Frequency uncertainty principle

$$\sigma_t \cdot \sigma_\omega \geq \frac{1}{2}$$



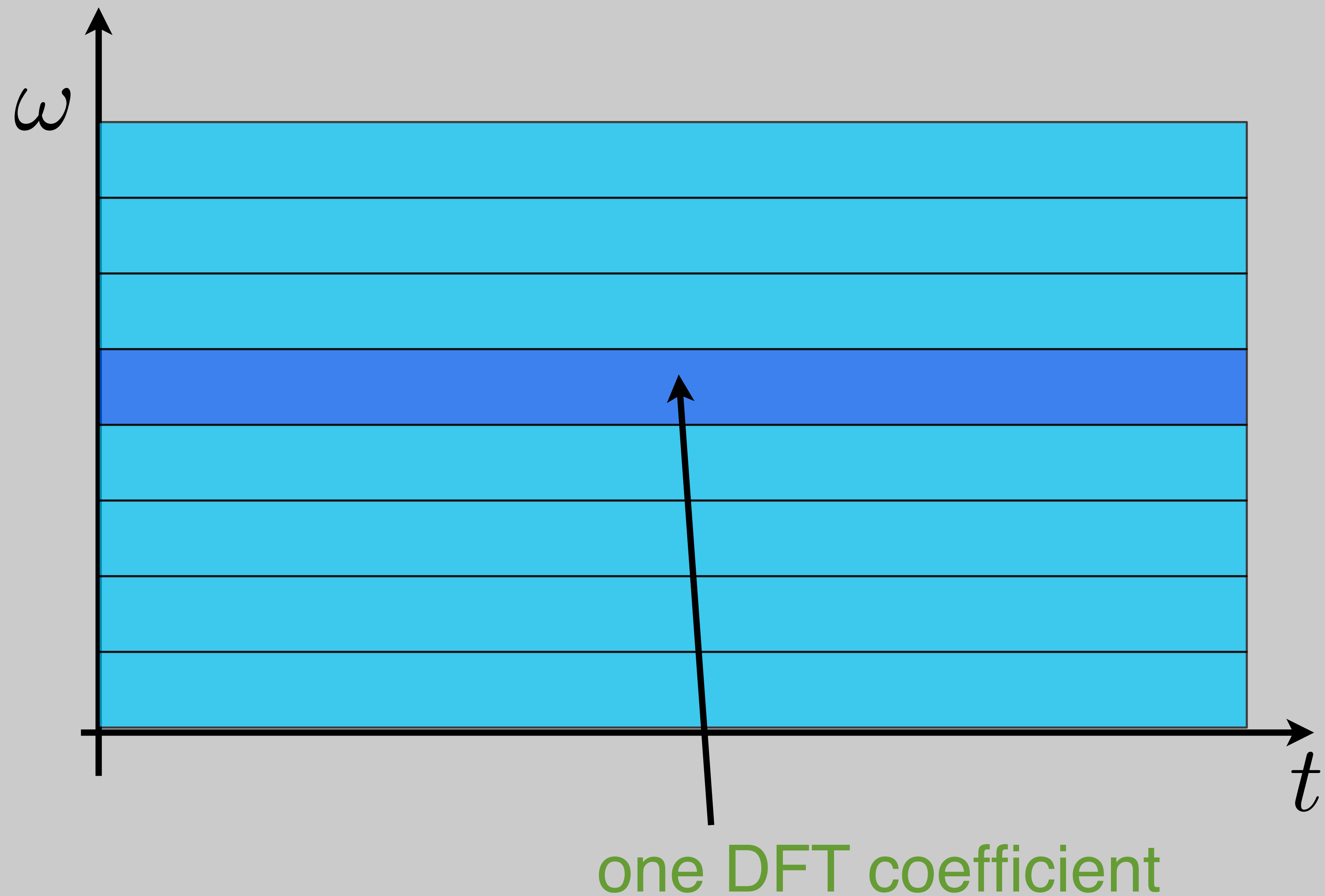
DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$\Delta\omega = \frac{2\pi}{N}$$

$$\Delta t = N$$

$$\Delta\omega \cdot \Delta t = 2\pi$$



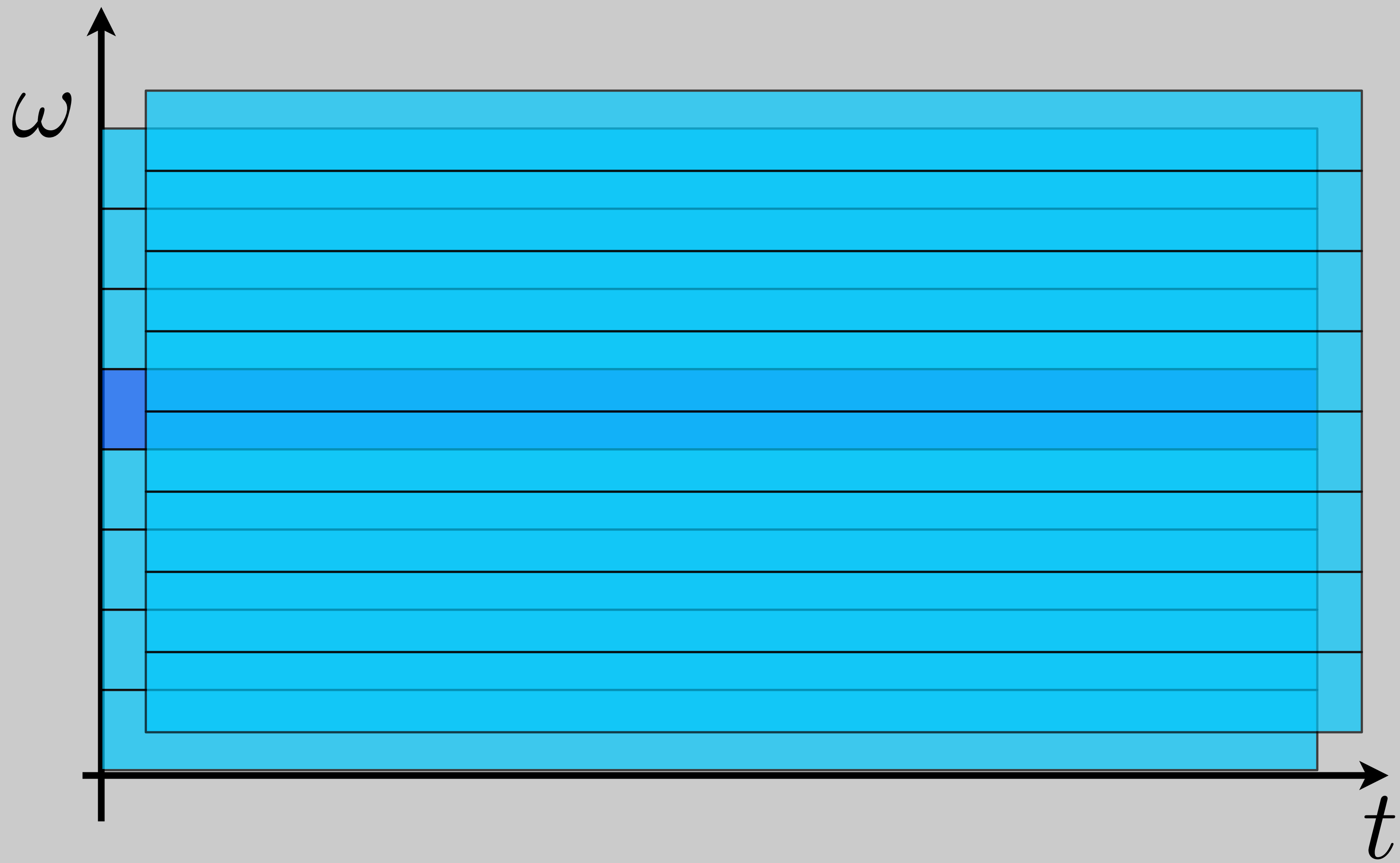
DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$\Delta\omega = \frac{2\pi}{N}$$

$$\Delta t = N$$

$$\Delta\omega \cdot \Delta t = 2\pi$$



Question: What is the effect of zero-padding?
Answer: Overlapped Tiling!

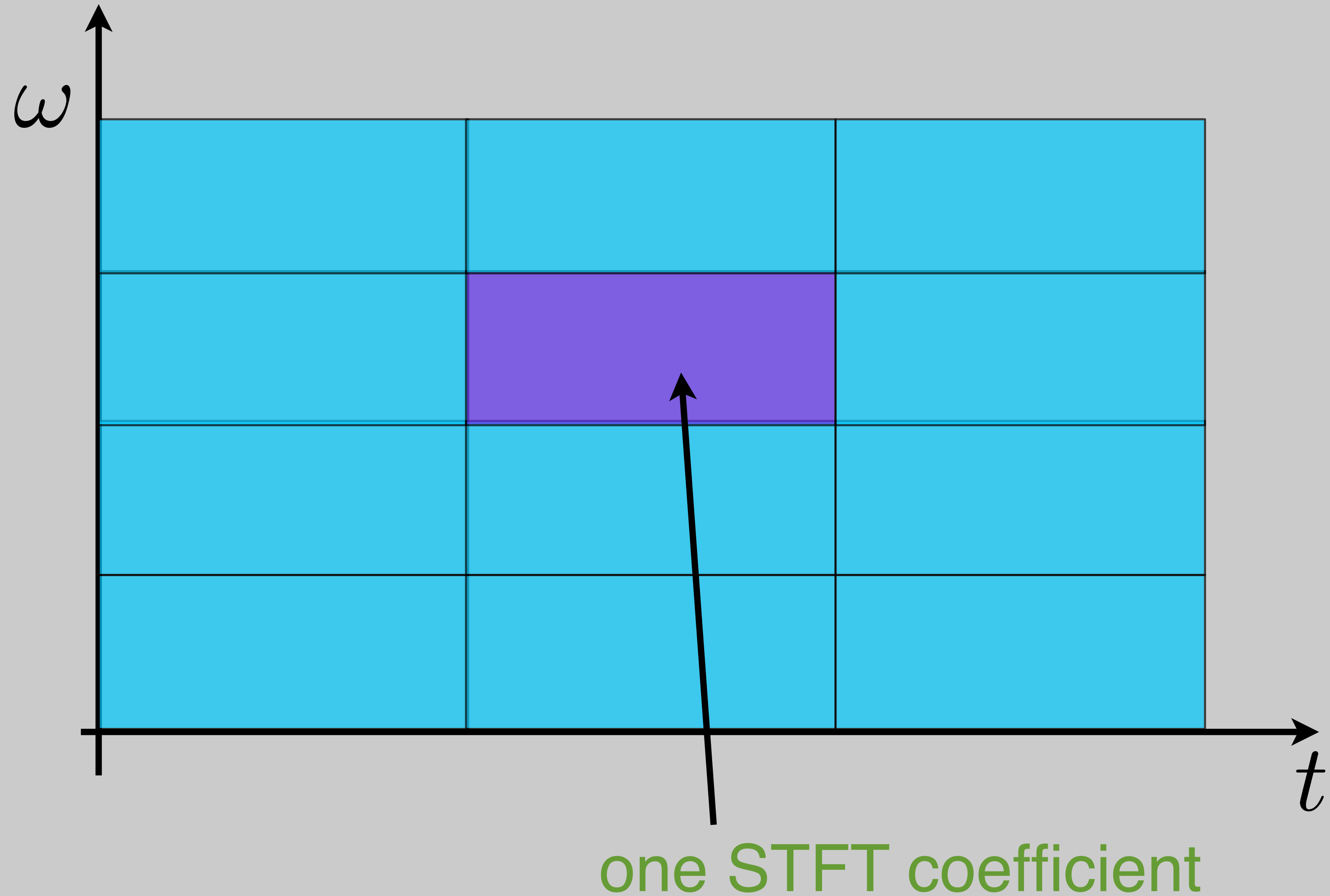
Discrete STFT

optional

$$X[r,k] = \sum_{m=0}^{L-1} x[r{\color{green}R} + m]w[m]e^{-j2\pi km/N}$$

$$\Delta\omega = \frac{2\pi}{L}$$

$$\Delta t = L$$



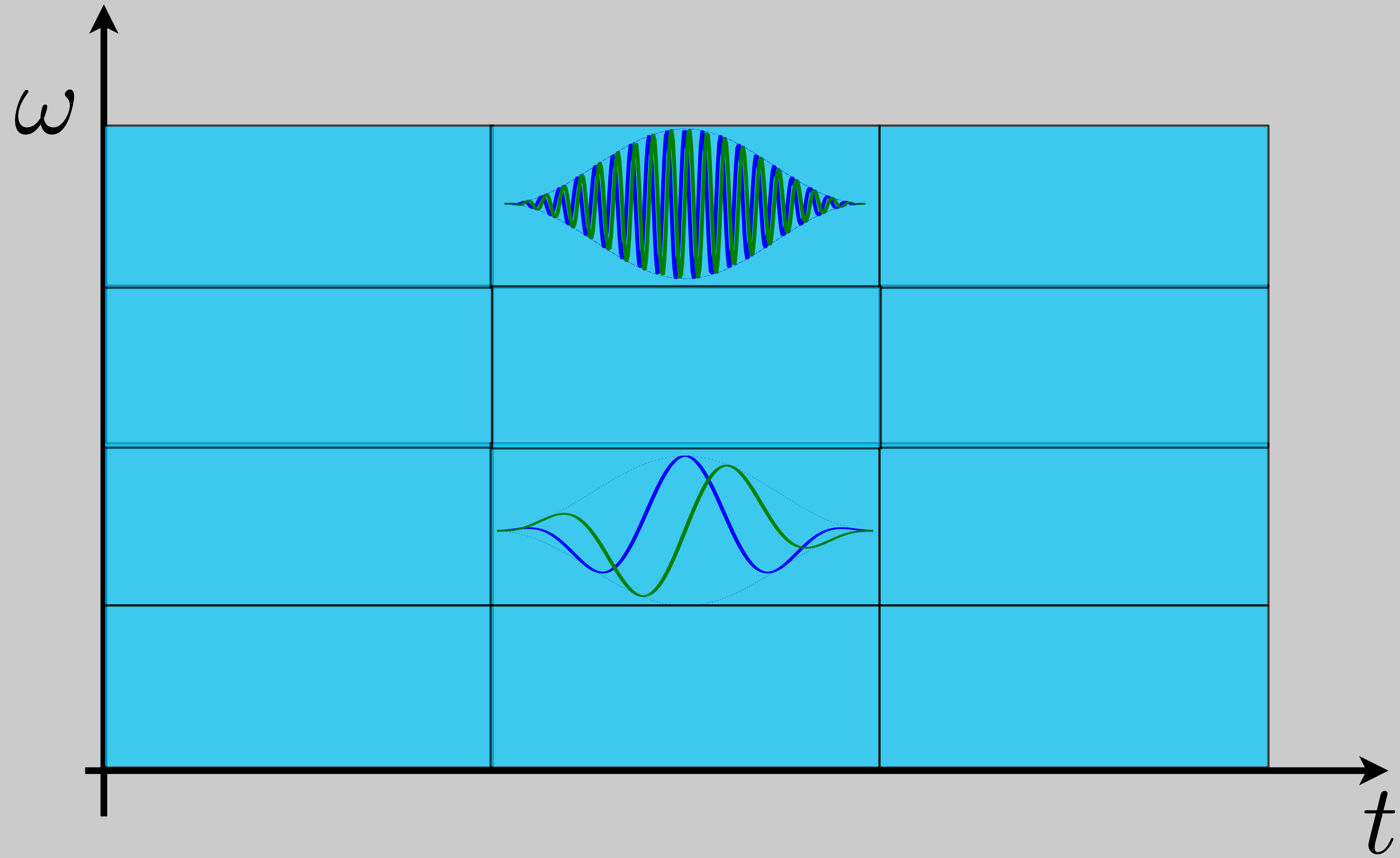
Discrete STFT

optional

$$X[r,k] = \sum_{m=0}^{L-1} x[r{\color{green}R}+m]w[m]e^{-j2\pi km/N}$$

$$\Delta\omega = \frac{2\pi}{L}$$

$$\Delta t = L$$



one STFT coefficient

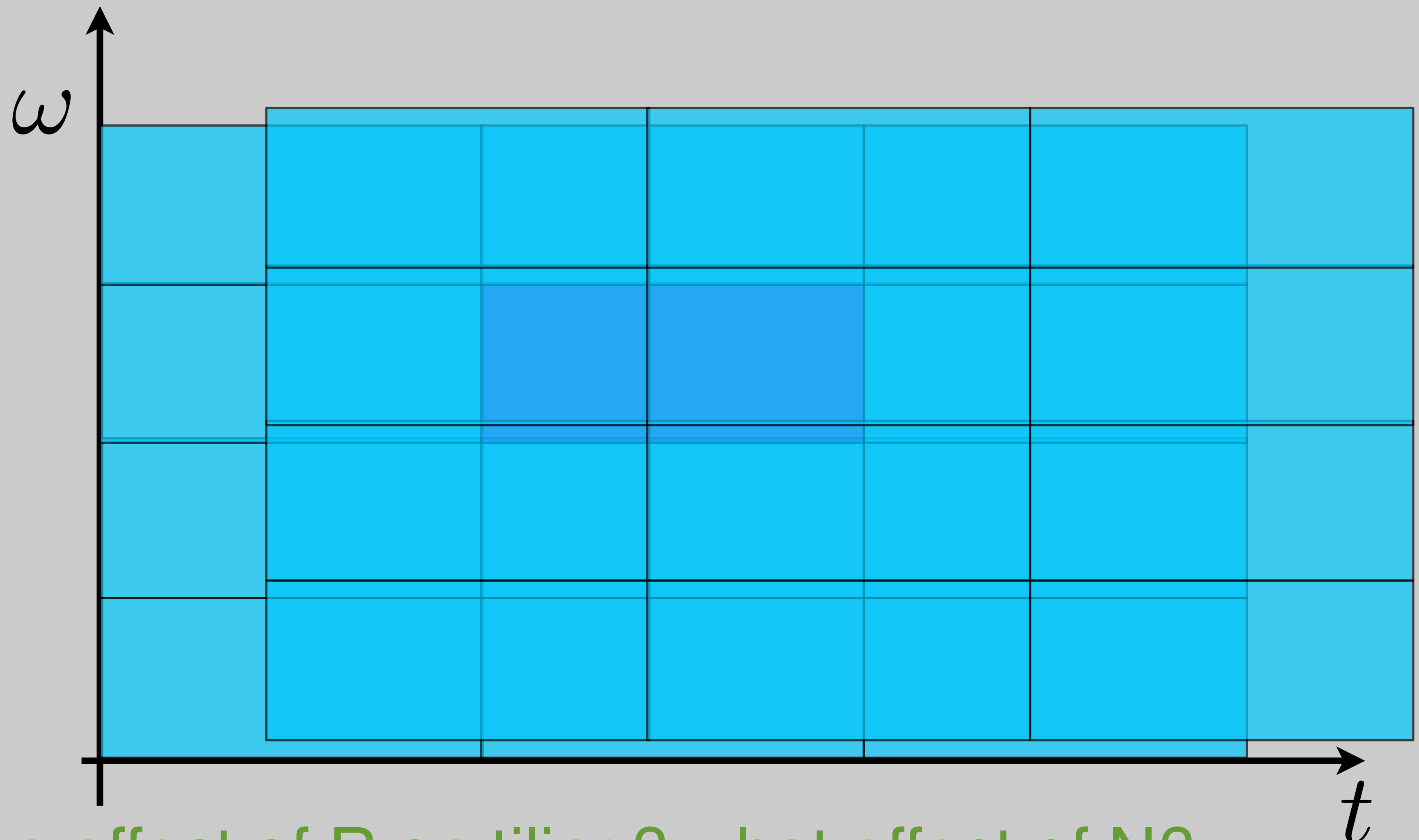
Discrete STFT

optional

$$X[r,k] = \sum_{m=0}^{L-1} x[r{\color{green}R} + m]w[m]e^{-j2\pi km/N}$$

$$\Delta\omega = \frac{2\pi}{L}$$

$$\Delta t = L$$

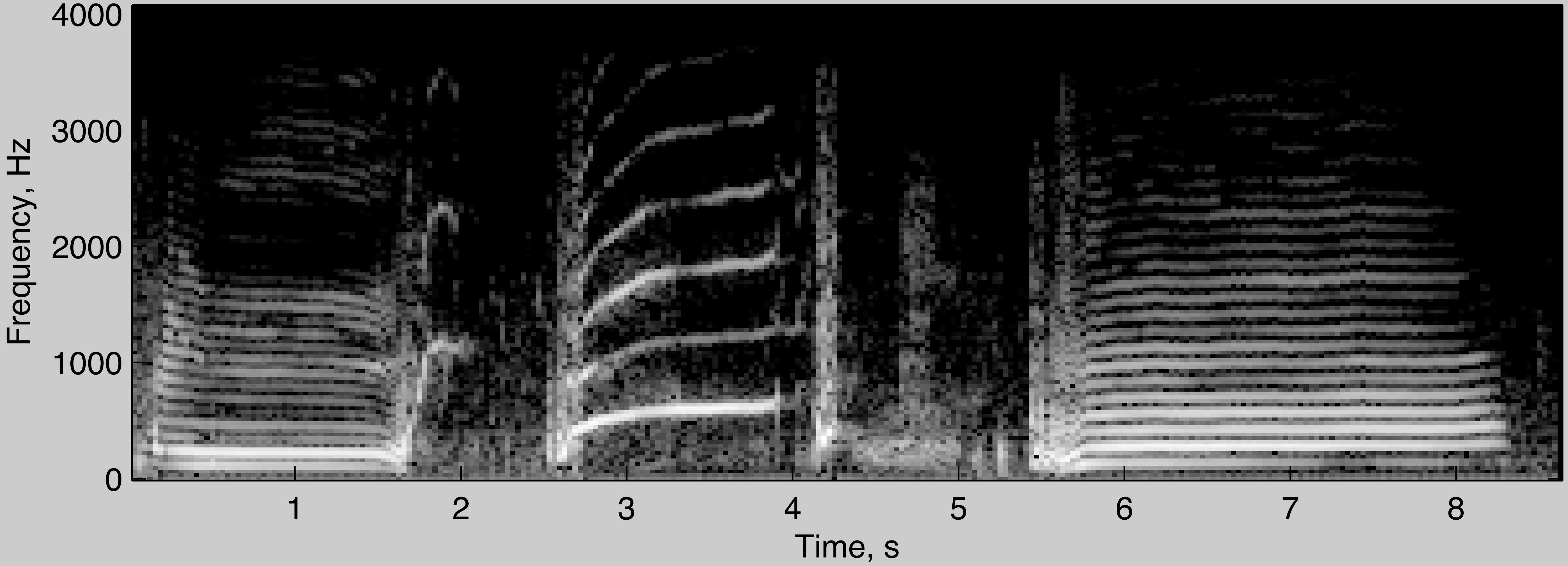


Question: What is the effect of R on tiling? what effect of N?
Answer: Overlapping in time or frequency or both!

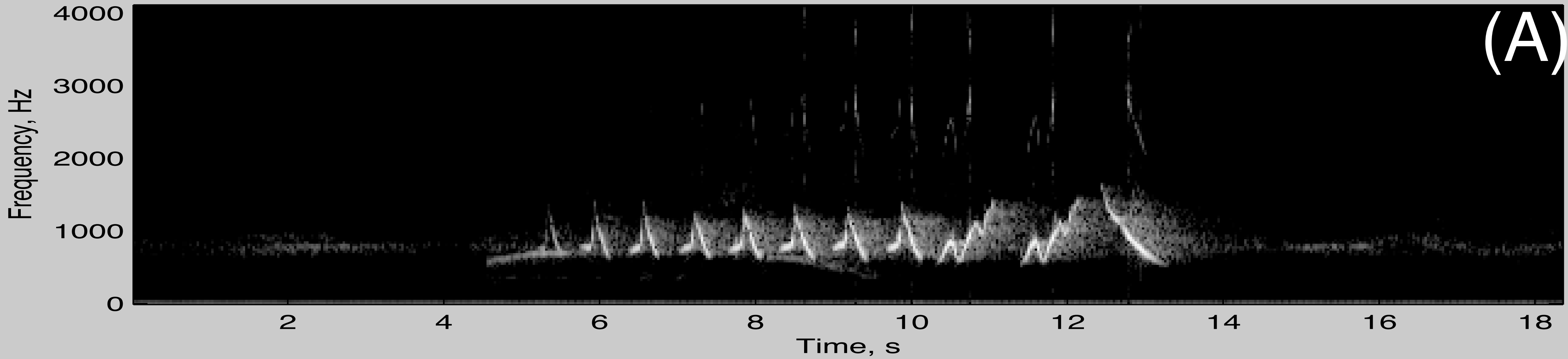
Applications

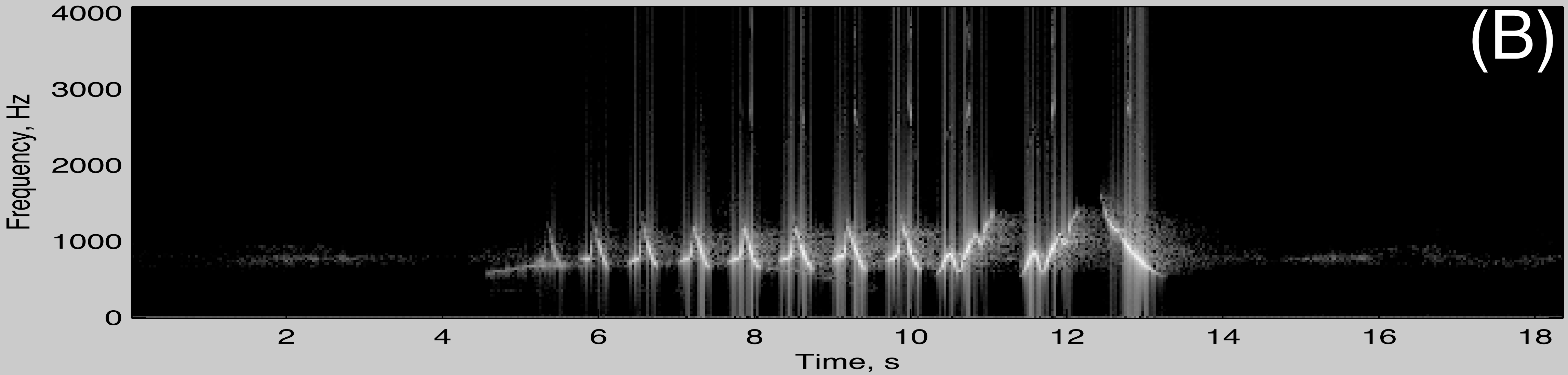
- Time Frequency Analysis

Spectrogram of Orca whale



Spectrogram





- What is the difference between the

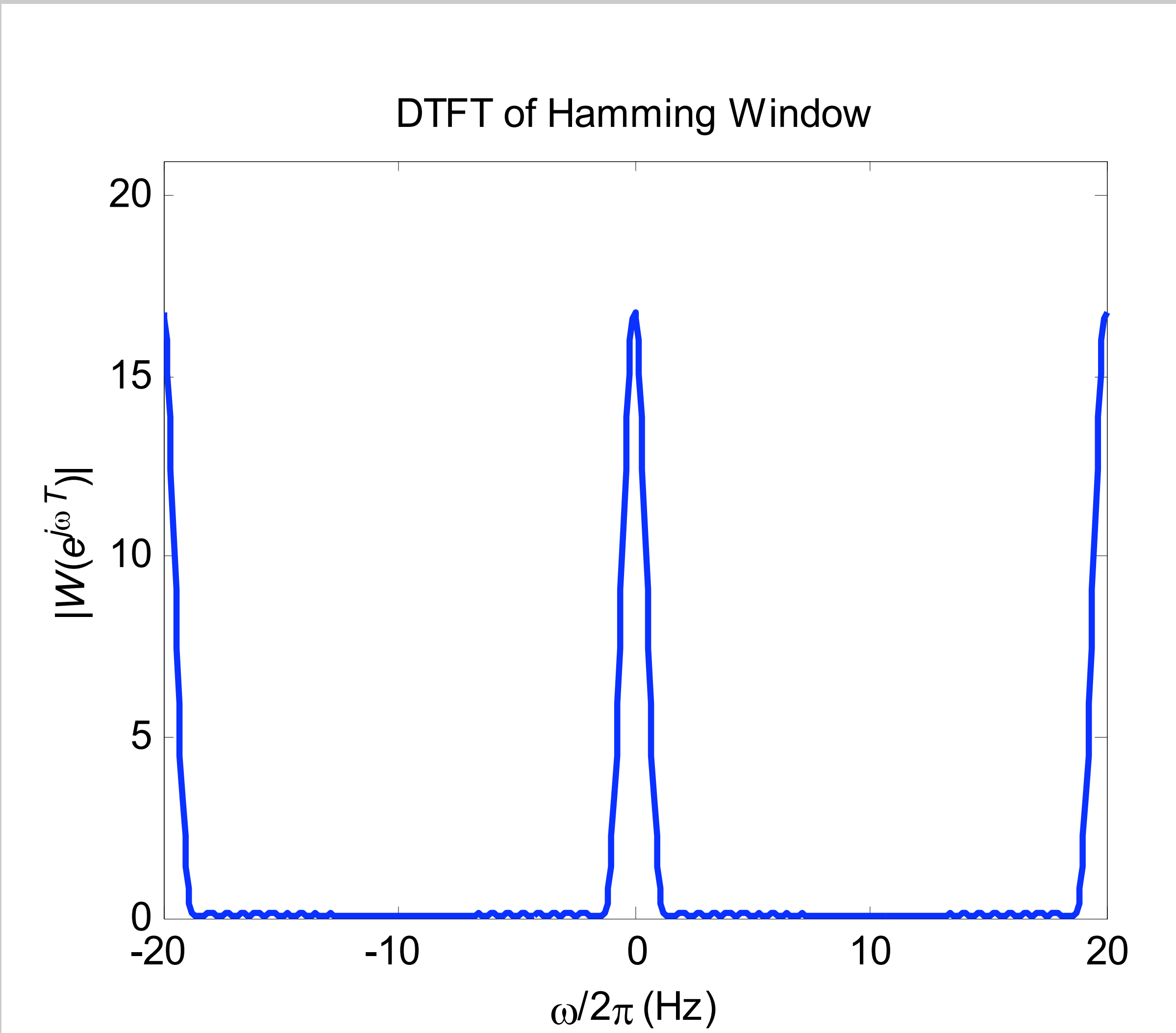
a) Window size B<A

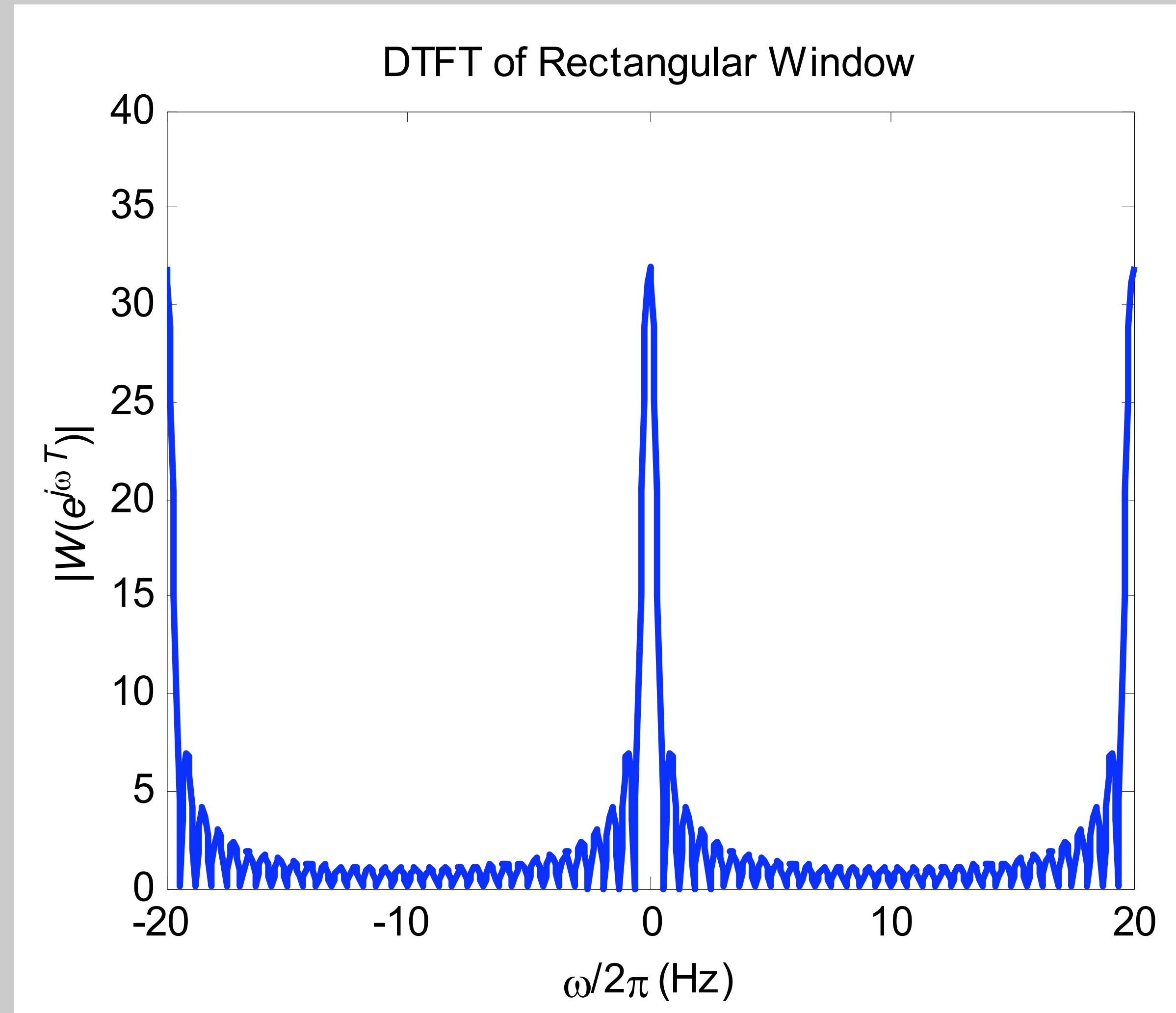
b) Window size B>A

c) Window type is different

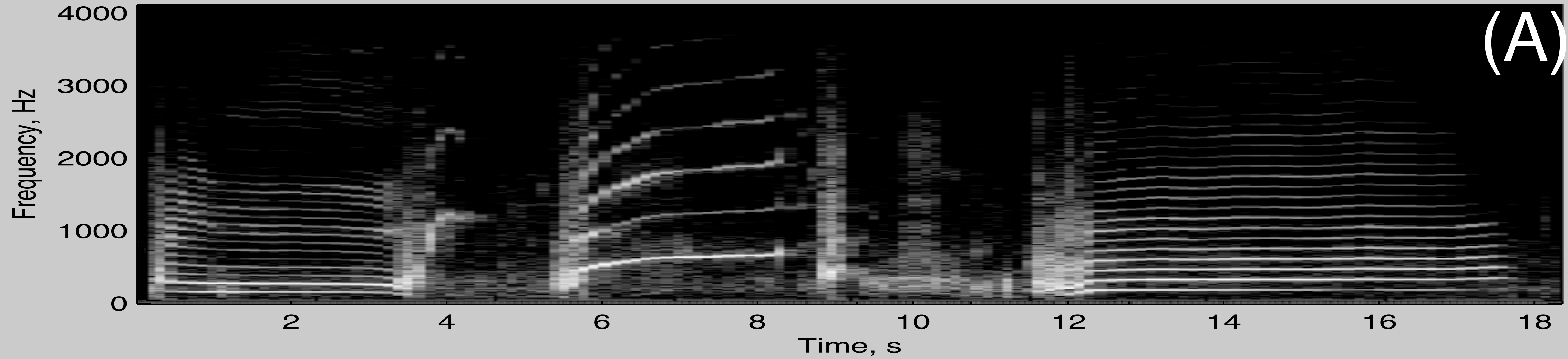
d) (A) uses overlapping window

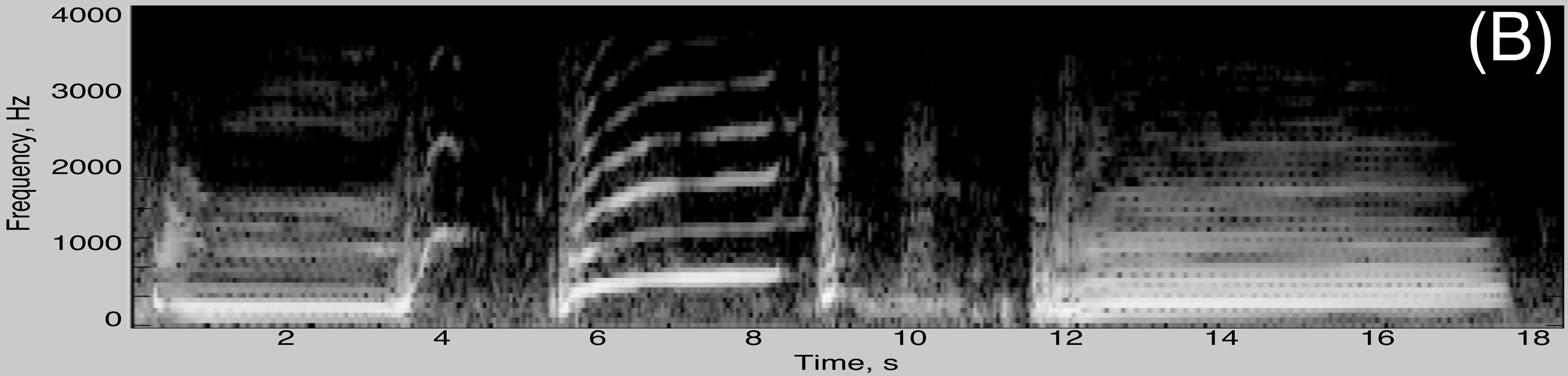
Sidelobes of Hann vs rectangular window





Spectrogram





- What is the difference between the

a) Window size B<A

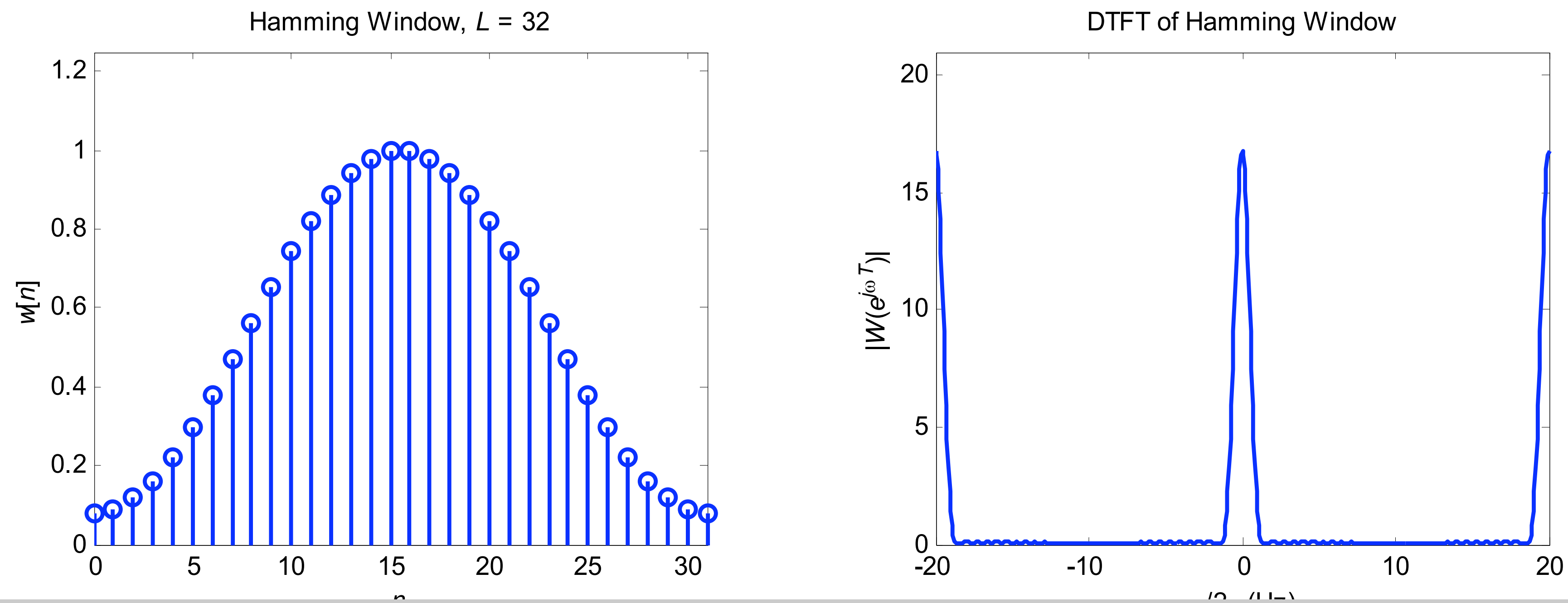
b) Window size B>A

c) Window type is different

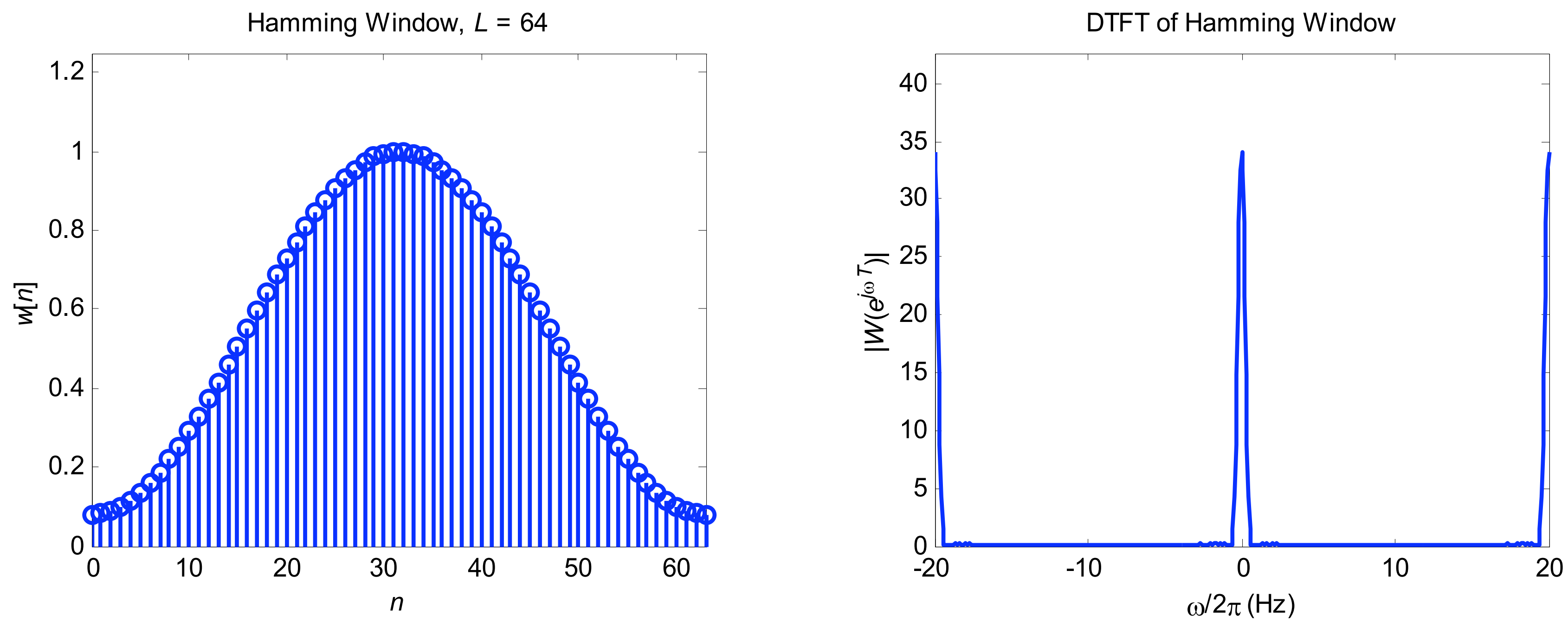
d) (A) uses overlapping window

Spectrogram

Hamming Window, $L = 32$



Hamming Window, $L = 64$

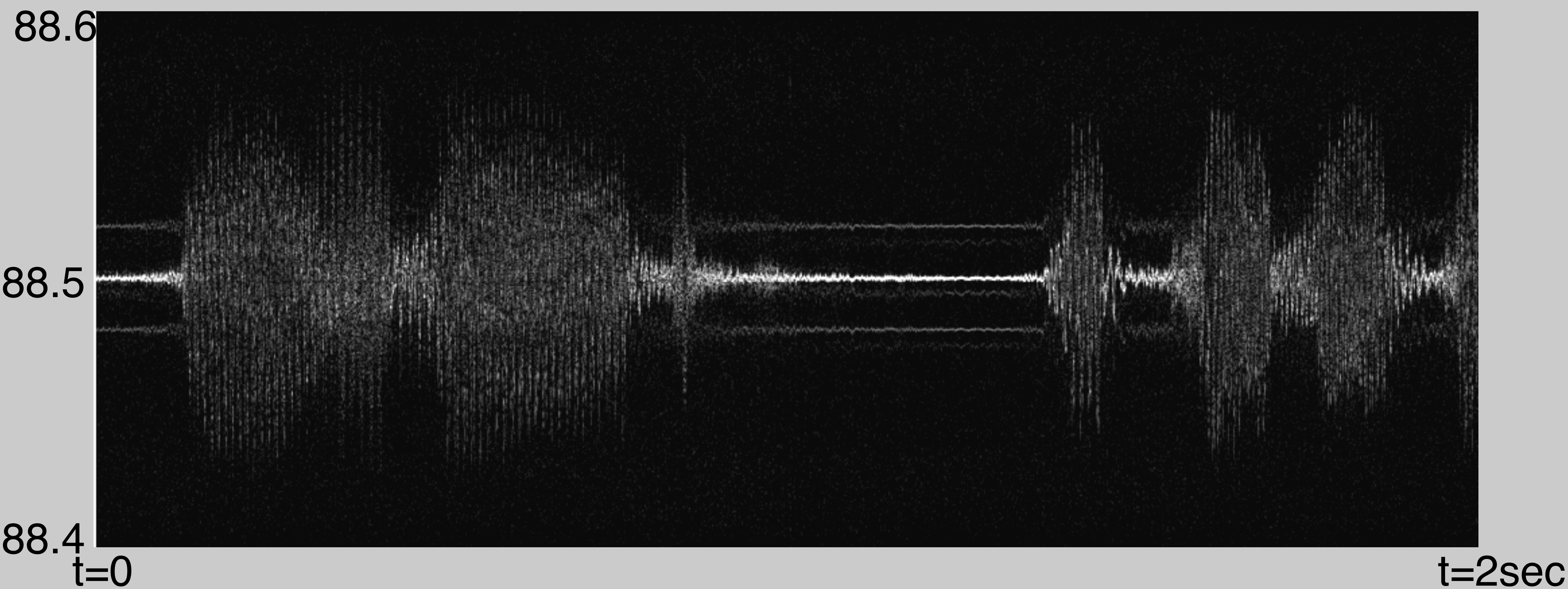


Spectrogram of FM

$$y_c(t) = A\cos\left(2\pi f_c t + 2\pi\Delta f \int_0^t x(\tau)d\tau\right)$$

$$y[n] = y(nT) = A\exp\left(j2\pi\Delta f \int_0^{nT} x(\tau)d\tau\right)$$

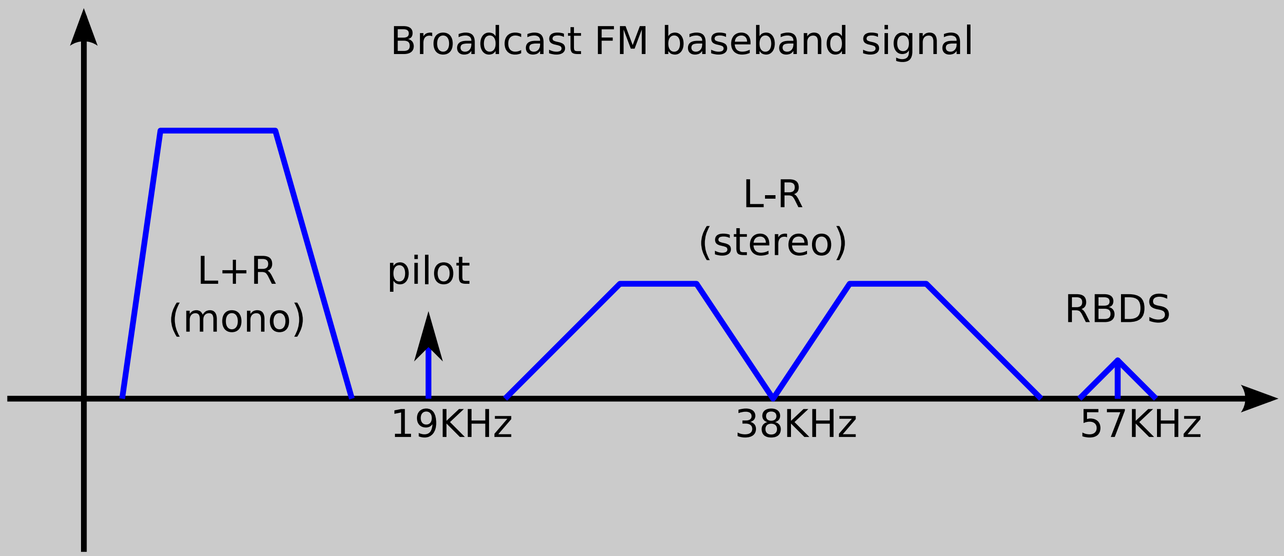
Spectrogram of FM radio



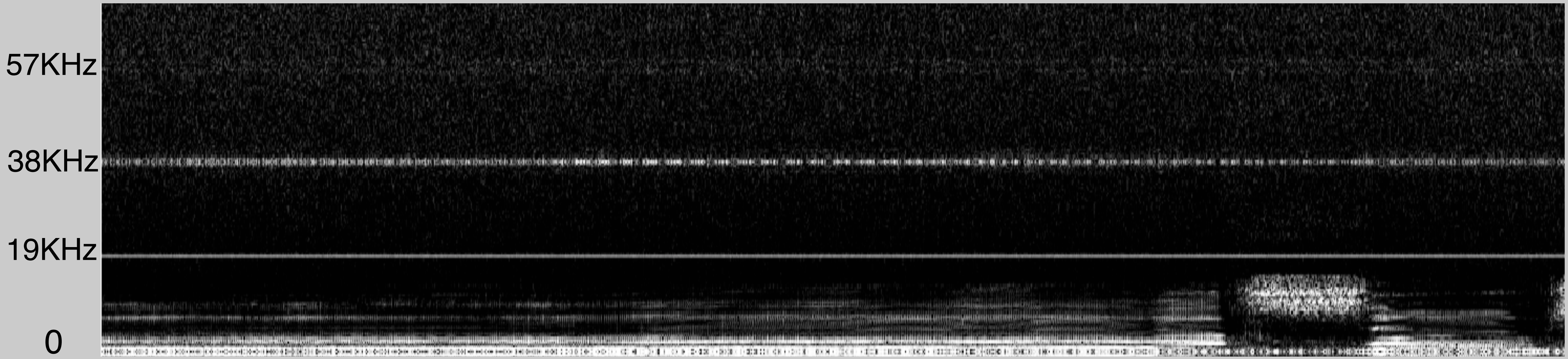
Spectrogram of FM radio Baseband

$$y[n] = y(nT) = A\exp\left(j2\pi\Delta f\int_0^{nT} x(\tau)d\tau\right)$$

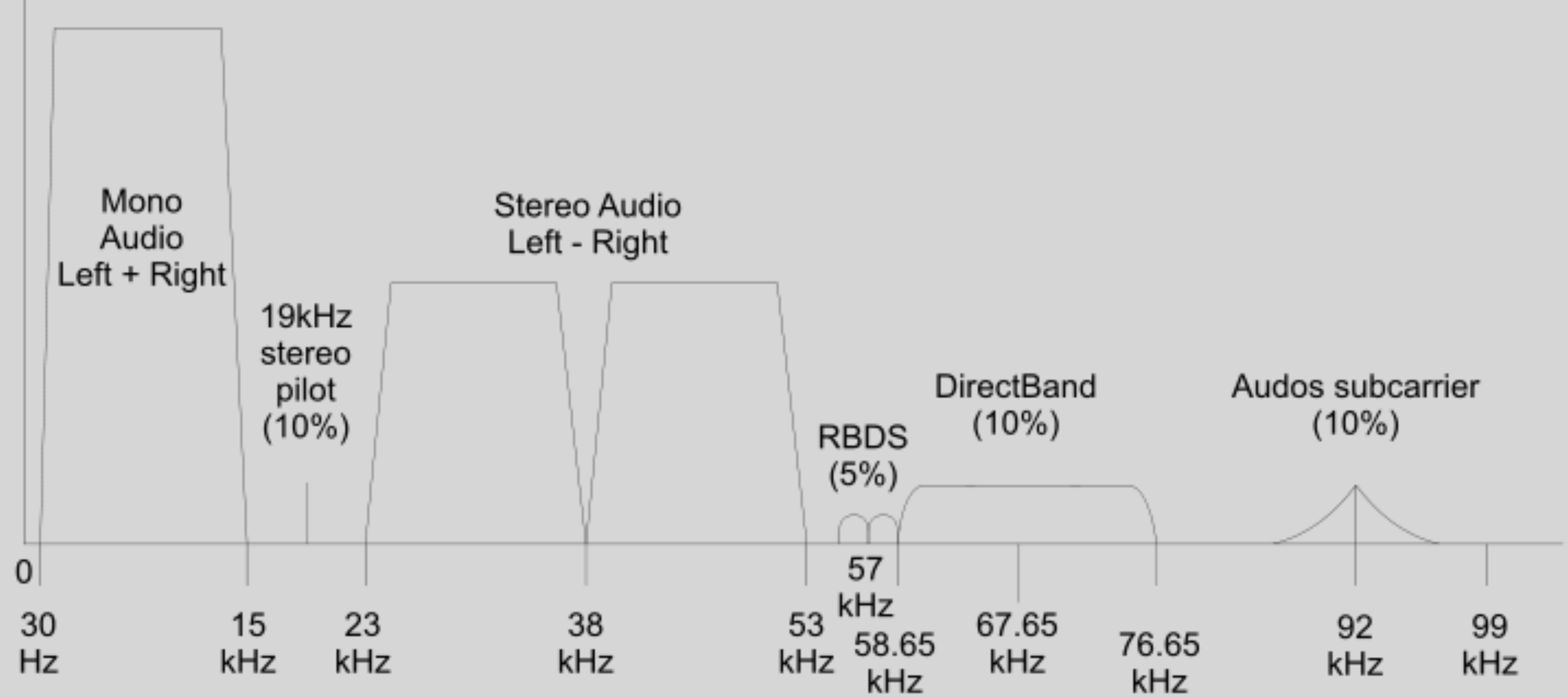
$$x(t) = \underbrace{(L+R)}_{\text{mono}} + \underbrace{0.1\cdot\cos(2\pi f_p t)}_{\text{pilot}} + \underbrace{(L-R)\cos(2\pi(2f_p)t)}_{\text{stereo}} + \underbrace{0.05\cdot \text{RBDS}(t)\cos(2\pi(3f_p)t)}_{\text{digital RBDS}}.$$



Spectrogram of **Demodulated** FM radio (Adele on 96.5 MHz)



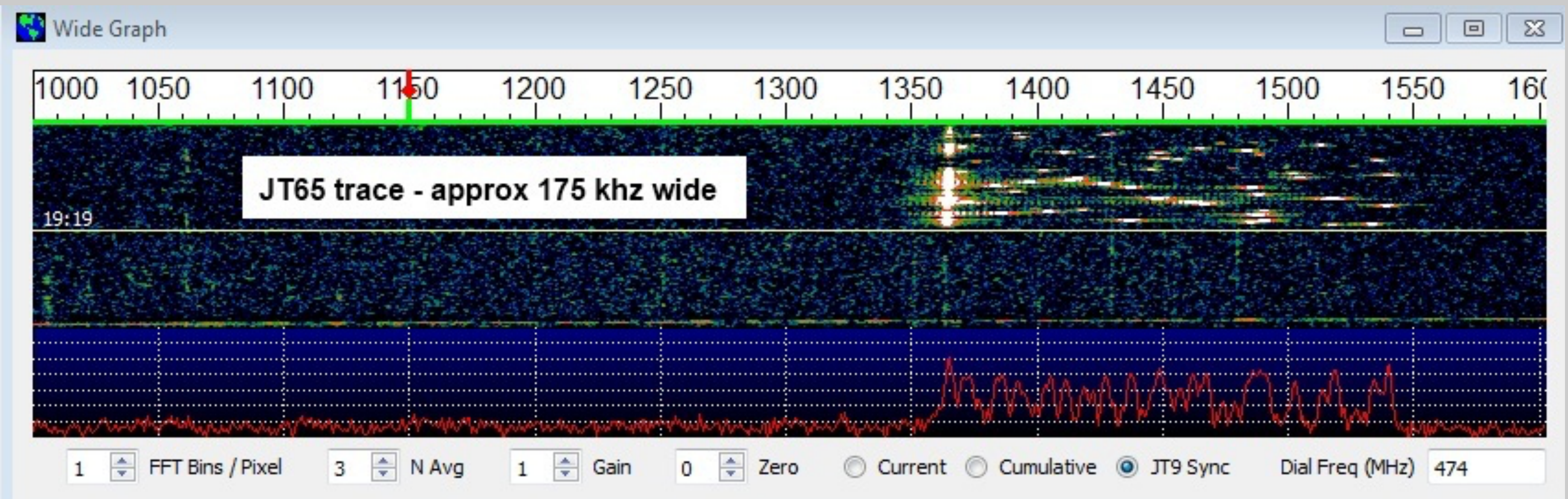
Subcarrier FM radio (Hidden Radio Stations)

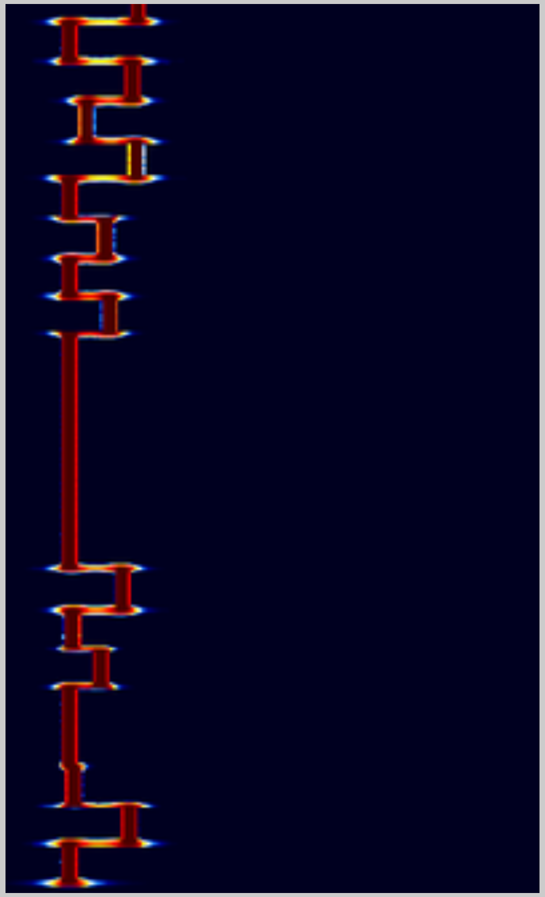


Applications

- Time Frequency Analysis

Spectrogram of digital communications -
Frequency Shift Keying JT65





https://gm7something.wordpress.com/2012/12/09/nov-radio-days/

Signal Wiki:
http://www.sigidwiki.com/wiki/Category:Active

STFT Reconstruction

$$x[rR+m]w_L[m] = \frac{1}{N}\sum_{k=0}^{N-1} X[n,k]e^{j2\pi km/N}$$

- For non-overlapping windows, R=L :

$$x[n] = \frac{x[n-rL]}{w_L[n-rL]}$$

$$rL \le n \le (r+1)R-1$$

- What is the problem?

STFT Reconstruction

$$x[rR+m]w_L[m] = \frac{1}{N}\sum_{k=0}^{N-1} X[n,k]e^{j2\pi km/N}$$

- For non-overlapping windows, R=L :

$$x[n] = \frac{x[n-rL]}{w_L[n-rL]}$$

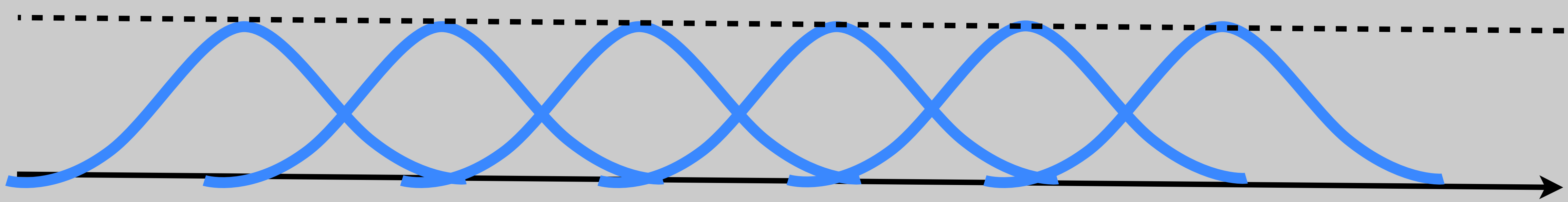
$$rL \le n \le (r+1)R - 1$$

- For stable reconstruction must overlap window 50% (at least)

STFT Reconstruction

- For stable reconstruction must overlap window 50% (at least)

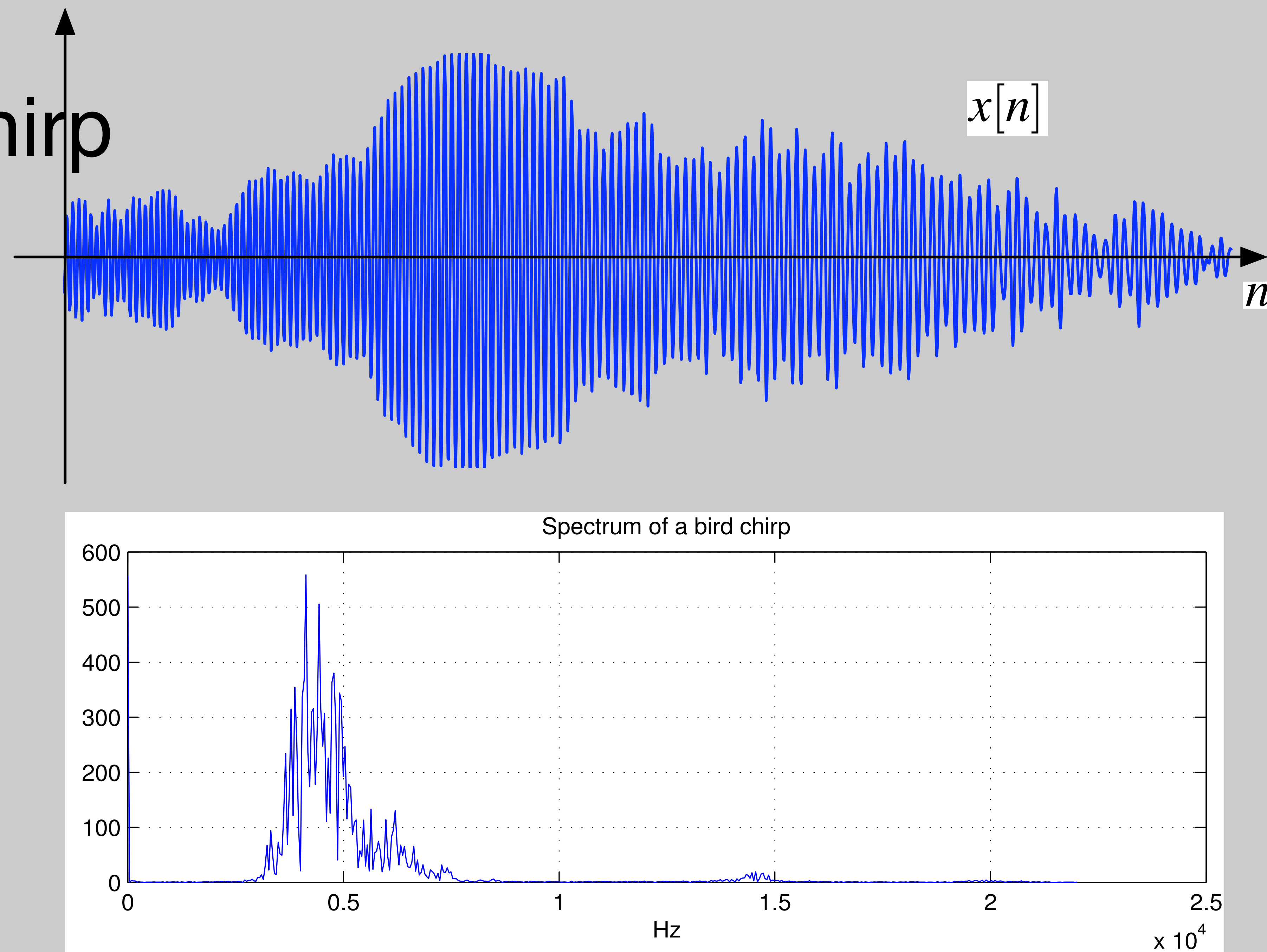
- For Hann, Bartlett reconstruct with overlap and add. No division!



Applications

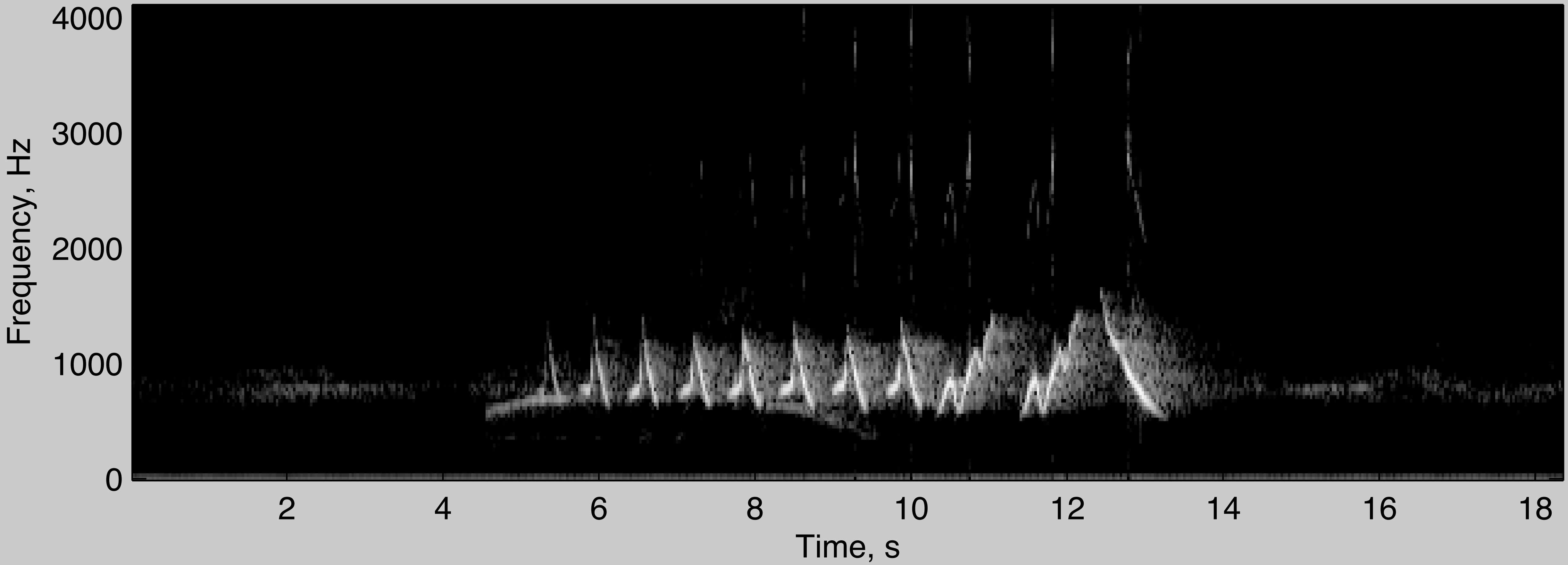
- Noise removal

- Recall bird chirp



Application

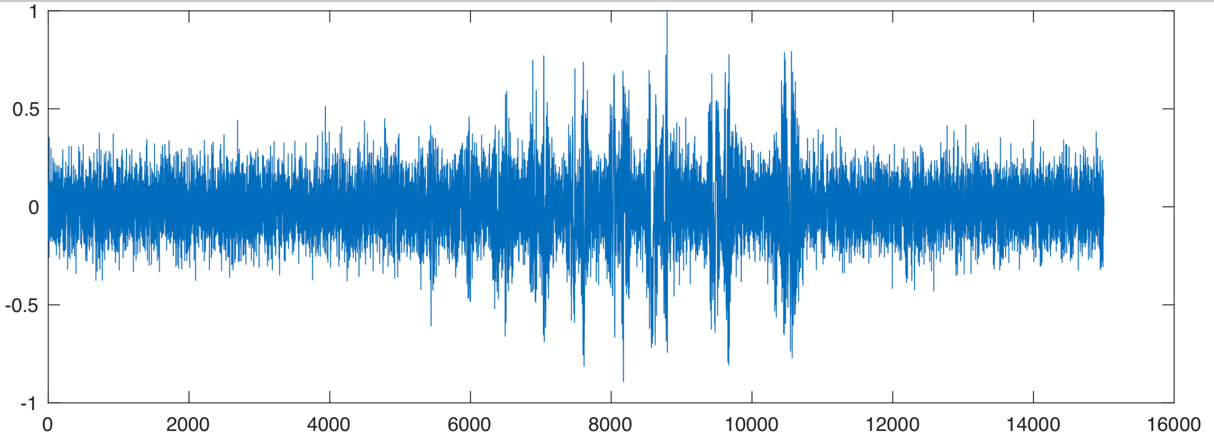
- Denoising of Sparse spectrograms

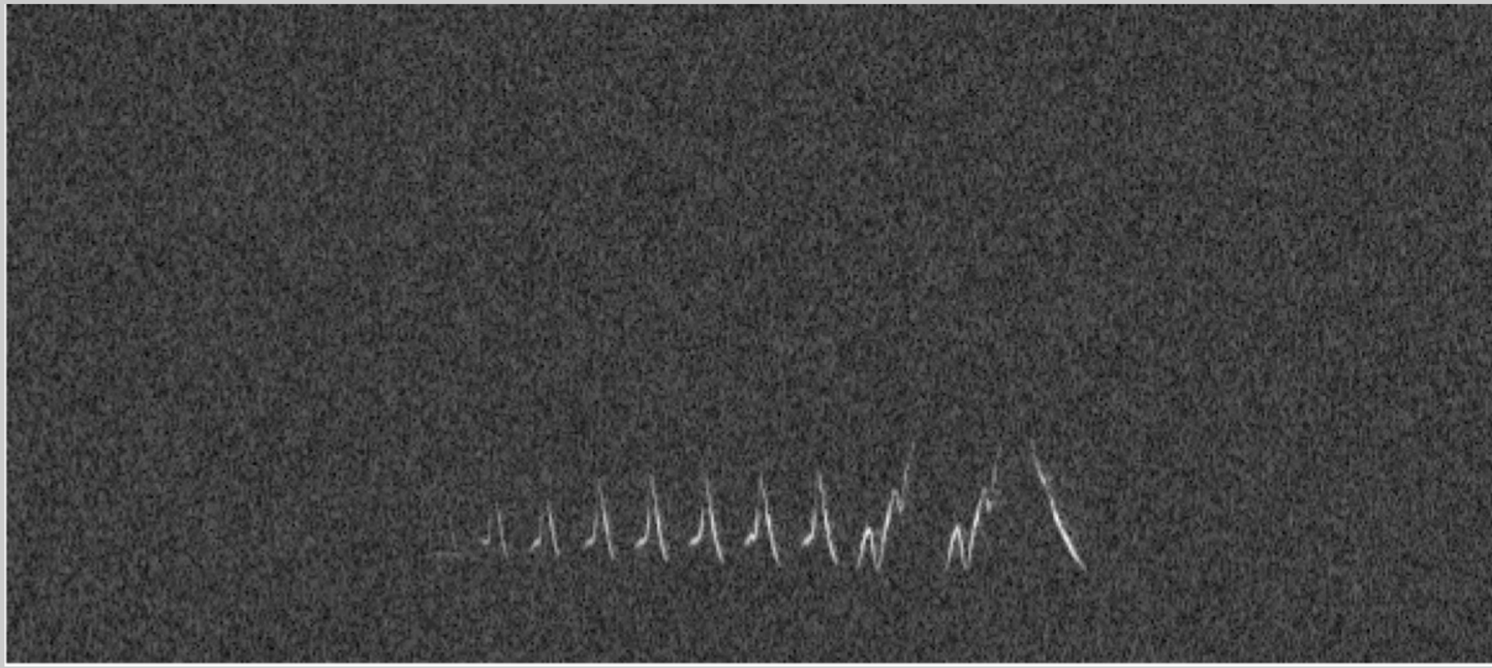


- Spectrum is sparse! can implement adaptive filter, or just threshold!

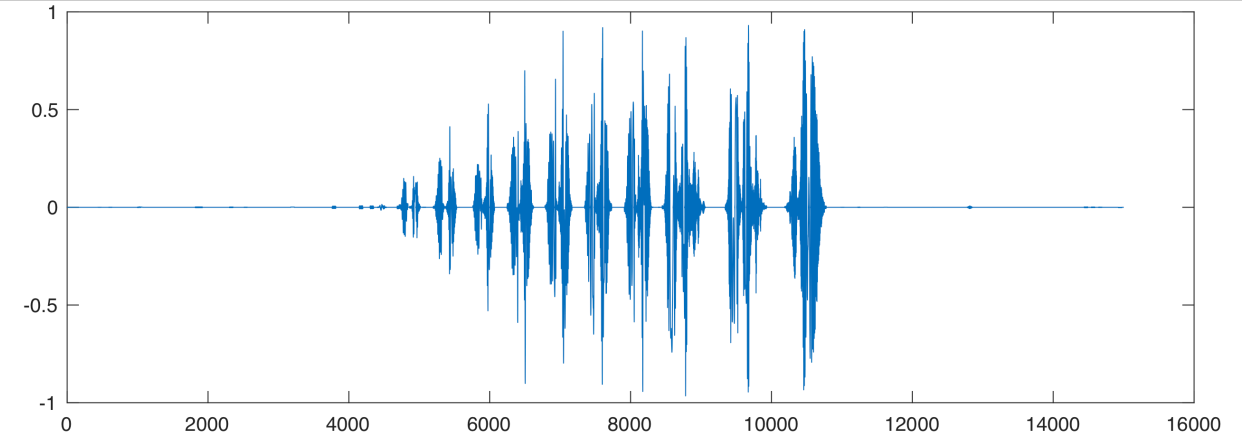
Denoising

Chirp with Gaussian Noise

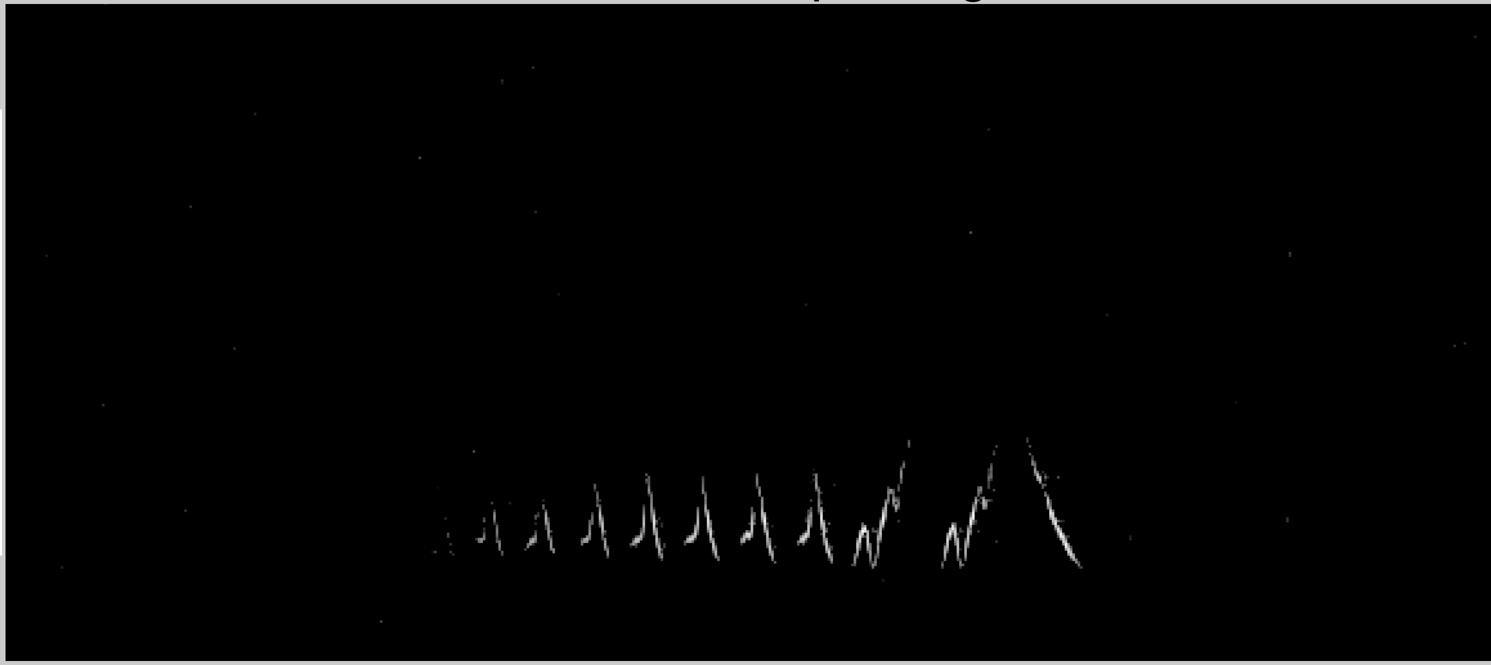




reconstructed time signal



Thresholded Spectrogram



Limitations of Discrete STFT

- Need overlapping $\Rightarrow$ Not orthogonal
- Computationally intensive O(MN log N)
- Same size Heisenberg boxes

- Basic Idea:
  - –low-freq changes slowly - fast tracking unimportant
  - –Fast tracking of high-freq is important in many apps.
  - –Must adapt Heisenberg box to frequency

- Back to continuous time for a bit.....

From STFT to Wavelets

- Continuous time