EE123
Digital Signal Processing

Lecture 5C
Introduction to Wavelets

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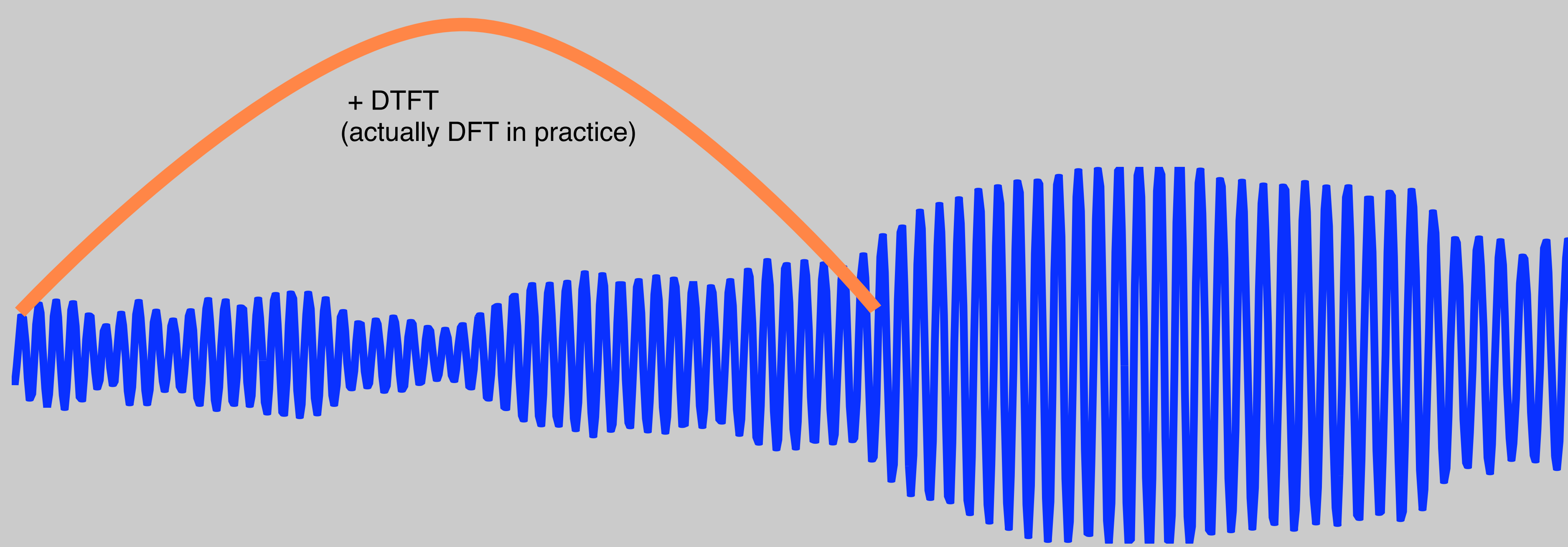
Time Dependent Fourier Transform

- To get temporal information, use part of the signal around every time point

$$X[n,\omega) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\omega m}$$

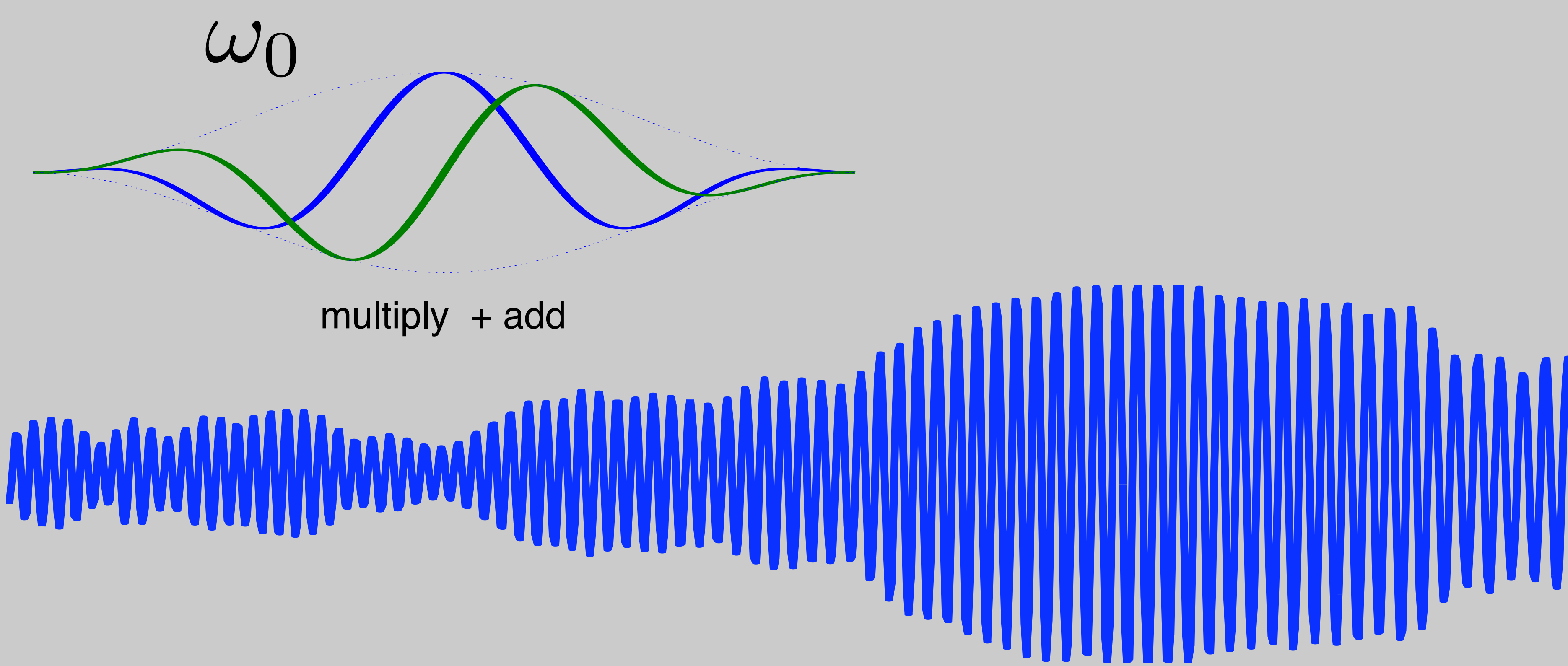
*Also called Short-time Fourier Transform (STFT)



Another view of STFT

- Can be expressed as a convolution

$$X[n,\omega) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\omega m}$$



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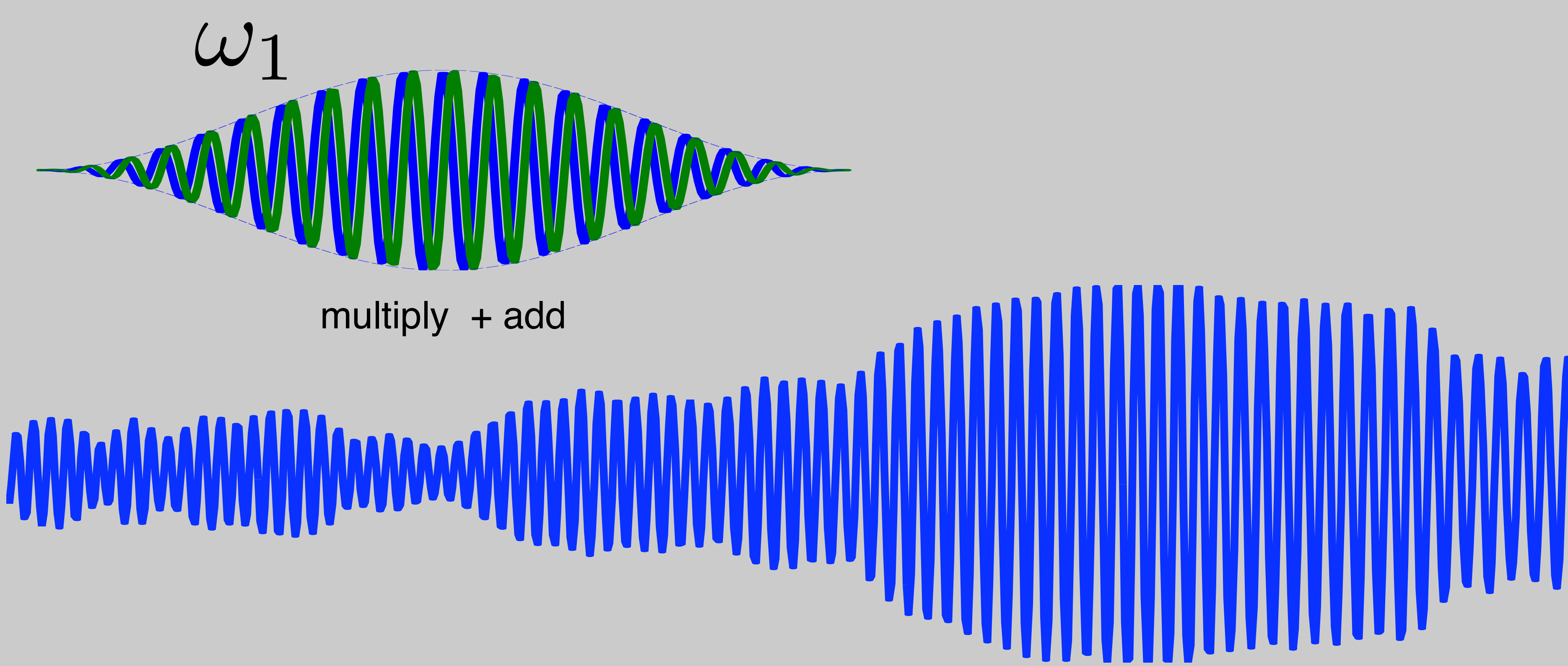
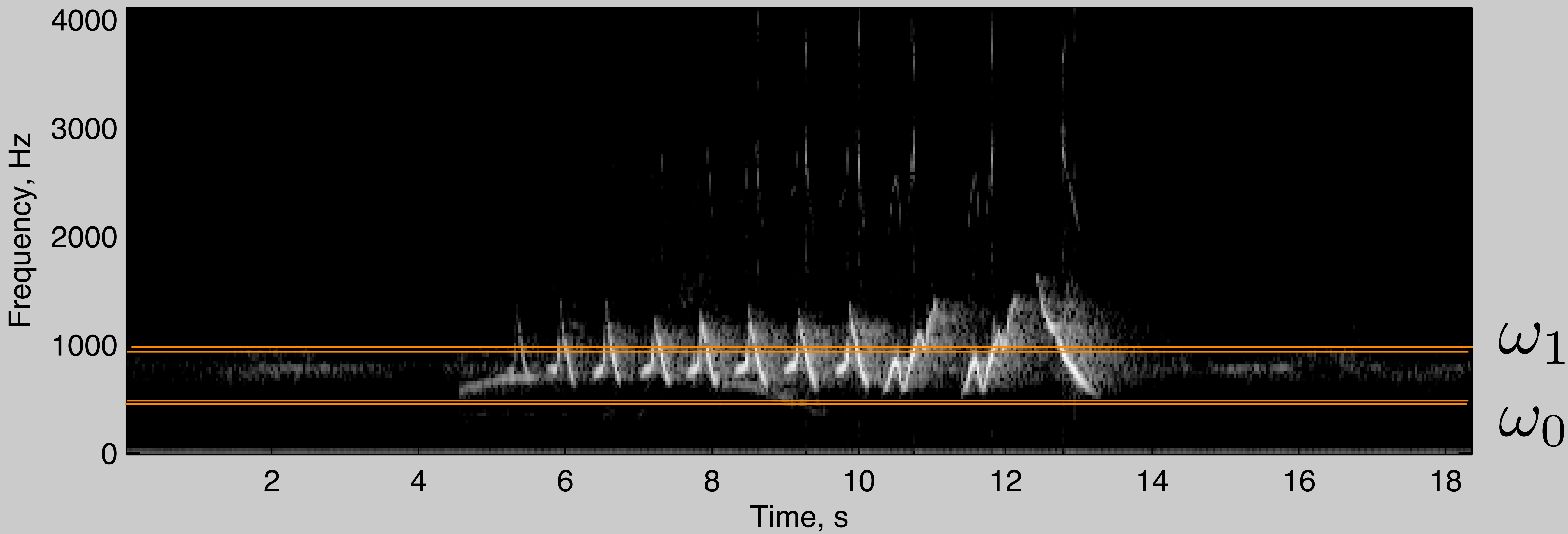
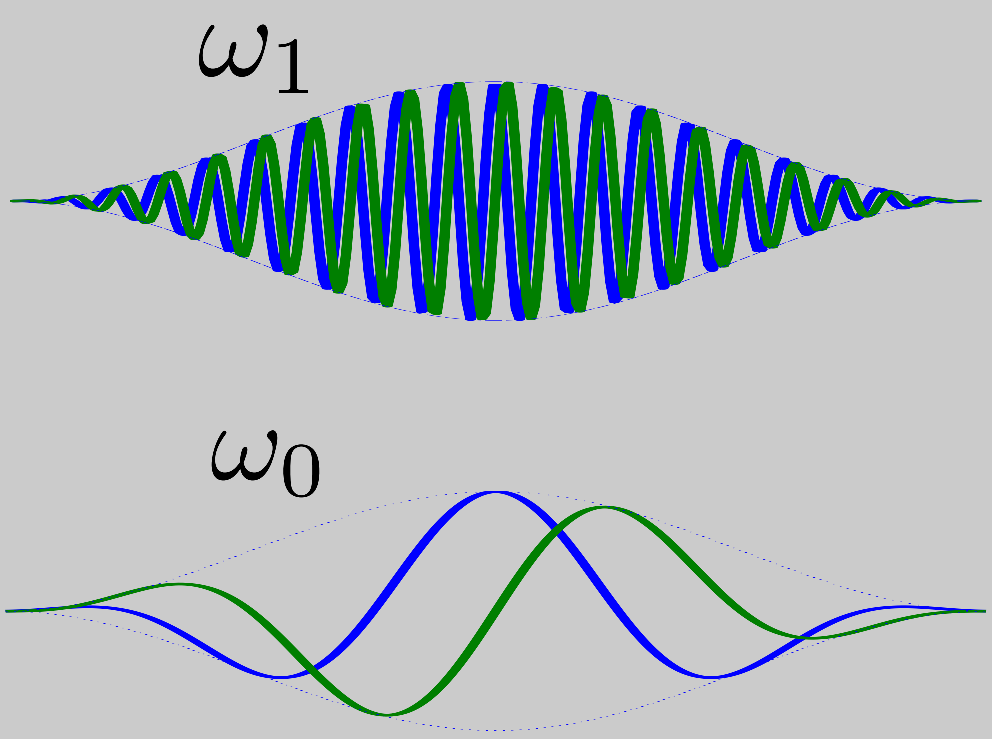
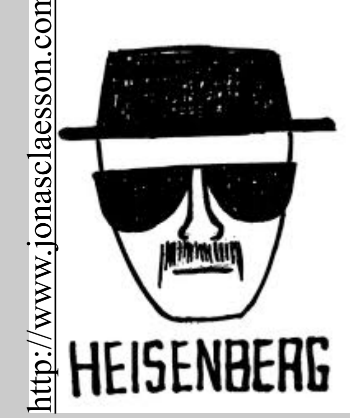


Basis functions (Atoms)

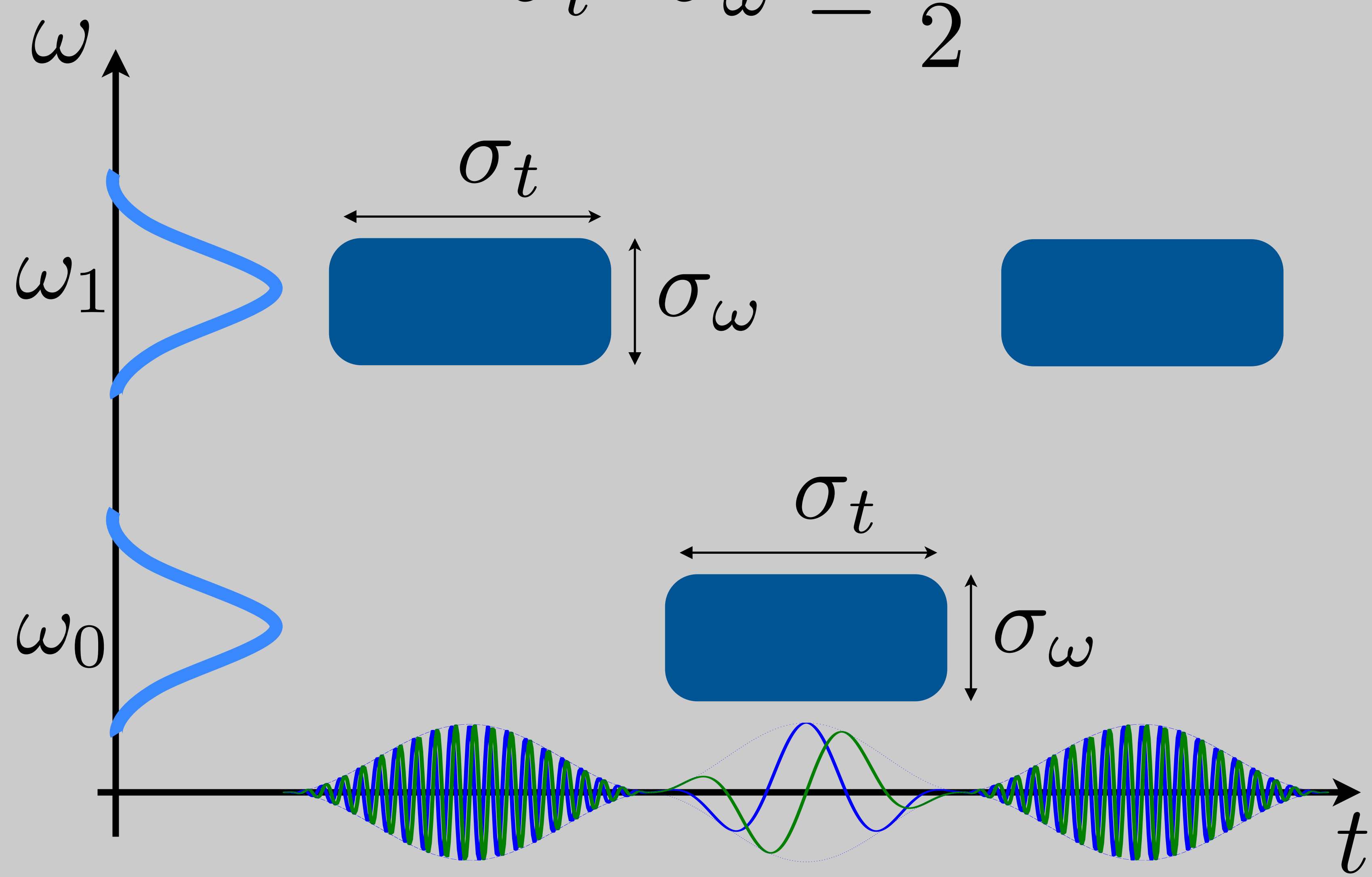






- Time-Frequency uncertainty principle

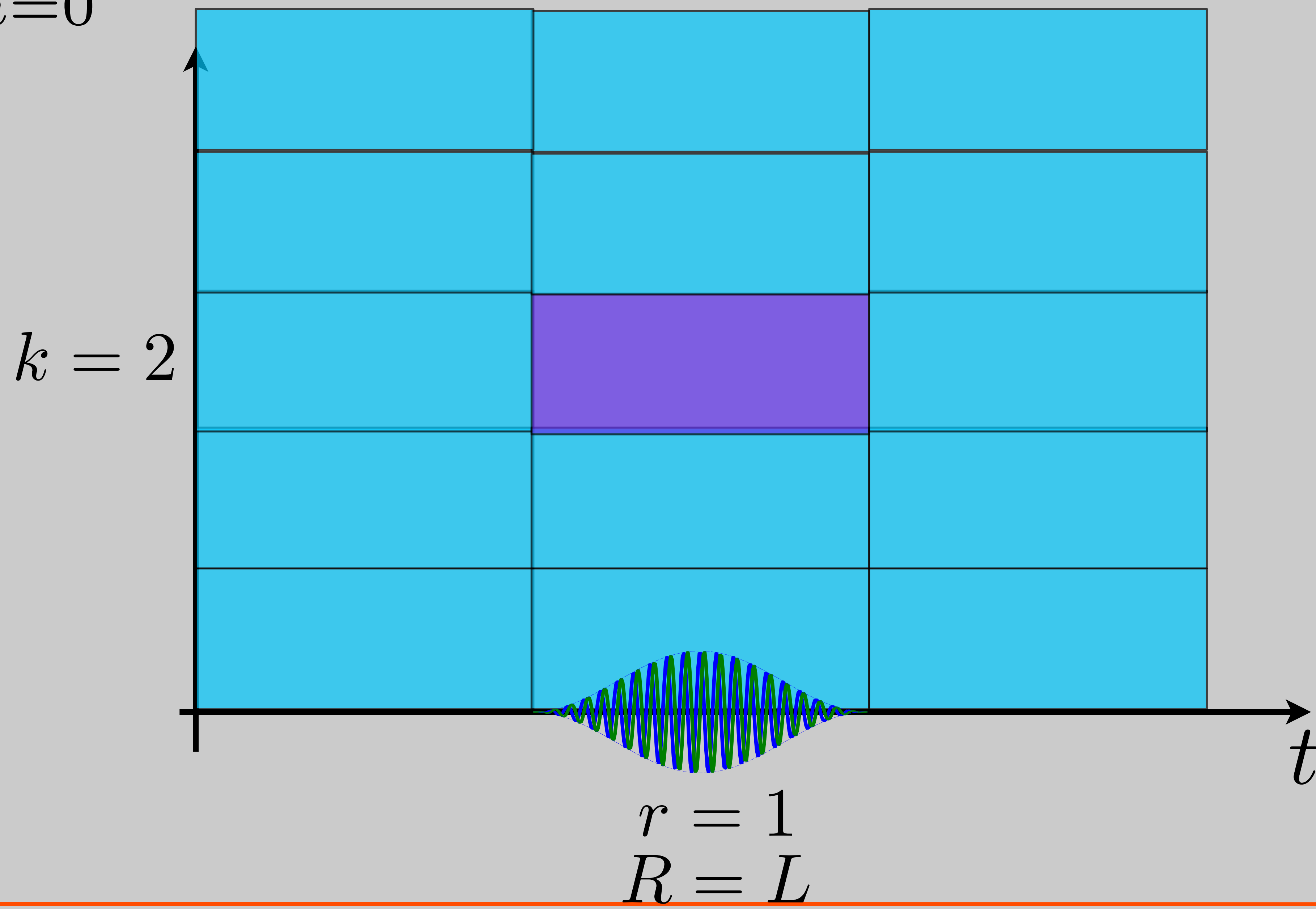
$$\sigma_t \cdot \sigma_\omega \geq \frac{1}{2}$$



Discrete STFT

optional

$$X[r,k] = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j2\pi km/N}$$



Discrete STFT

$$X[r,k] = \sum_{m=0}^{L-1} x[r\color{green}{R} + m]w[m]e^{-j2\pi km/N}$$

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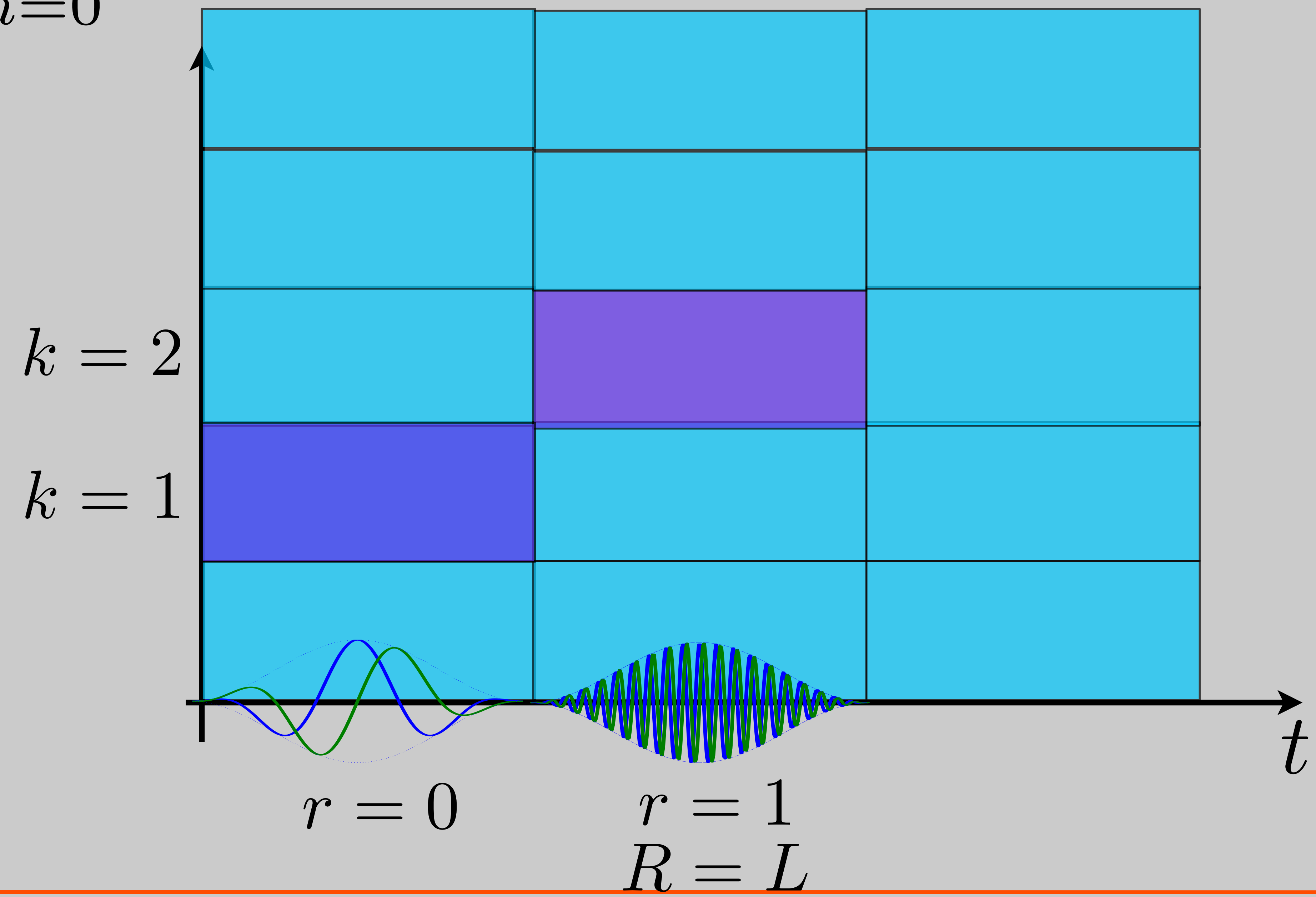

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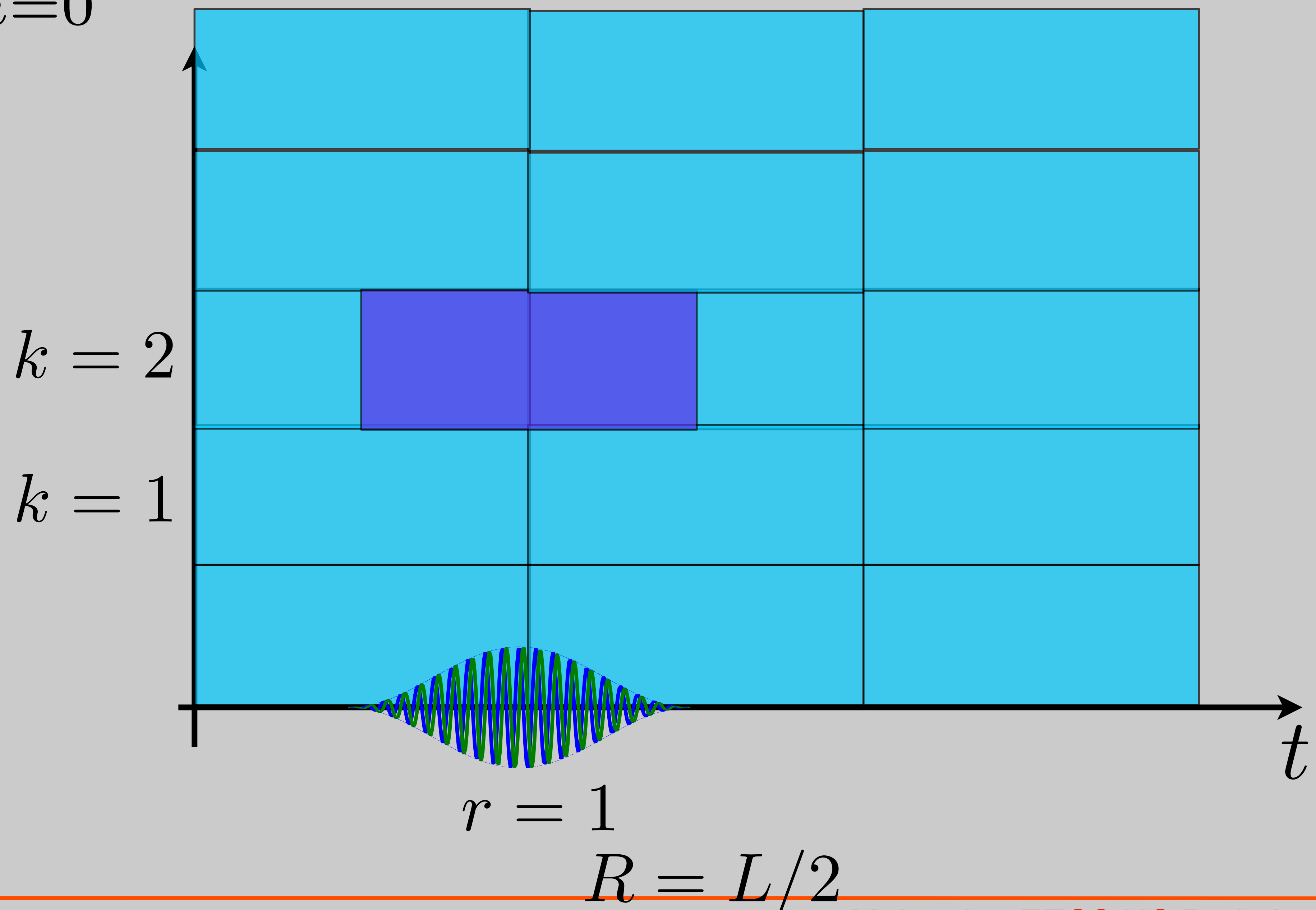

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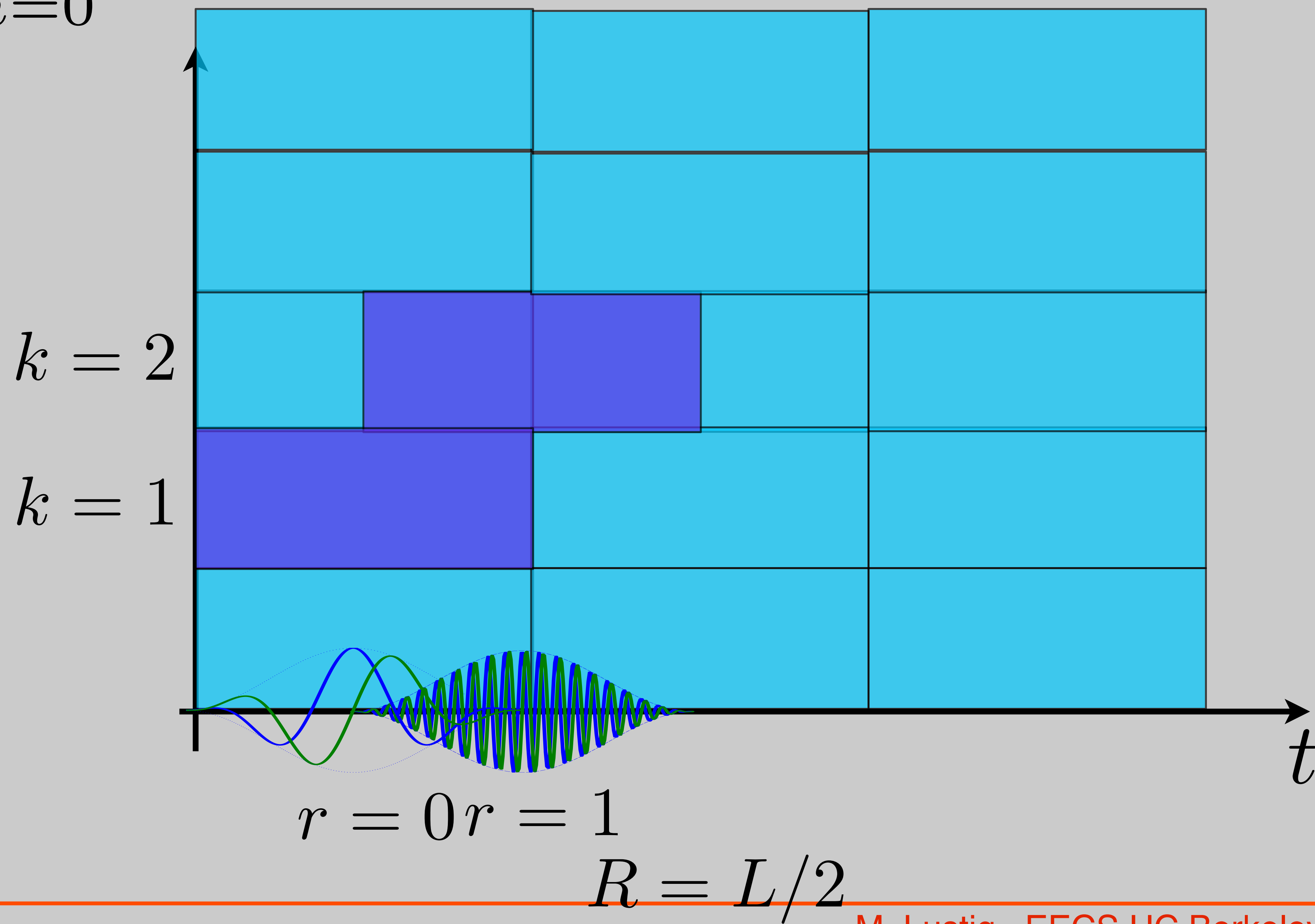


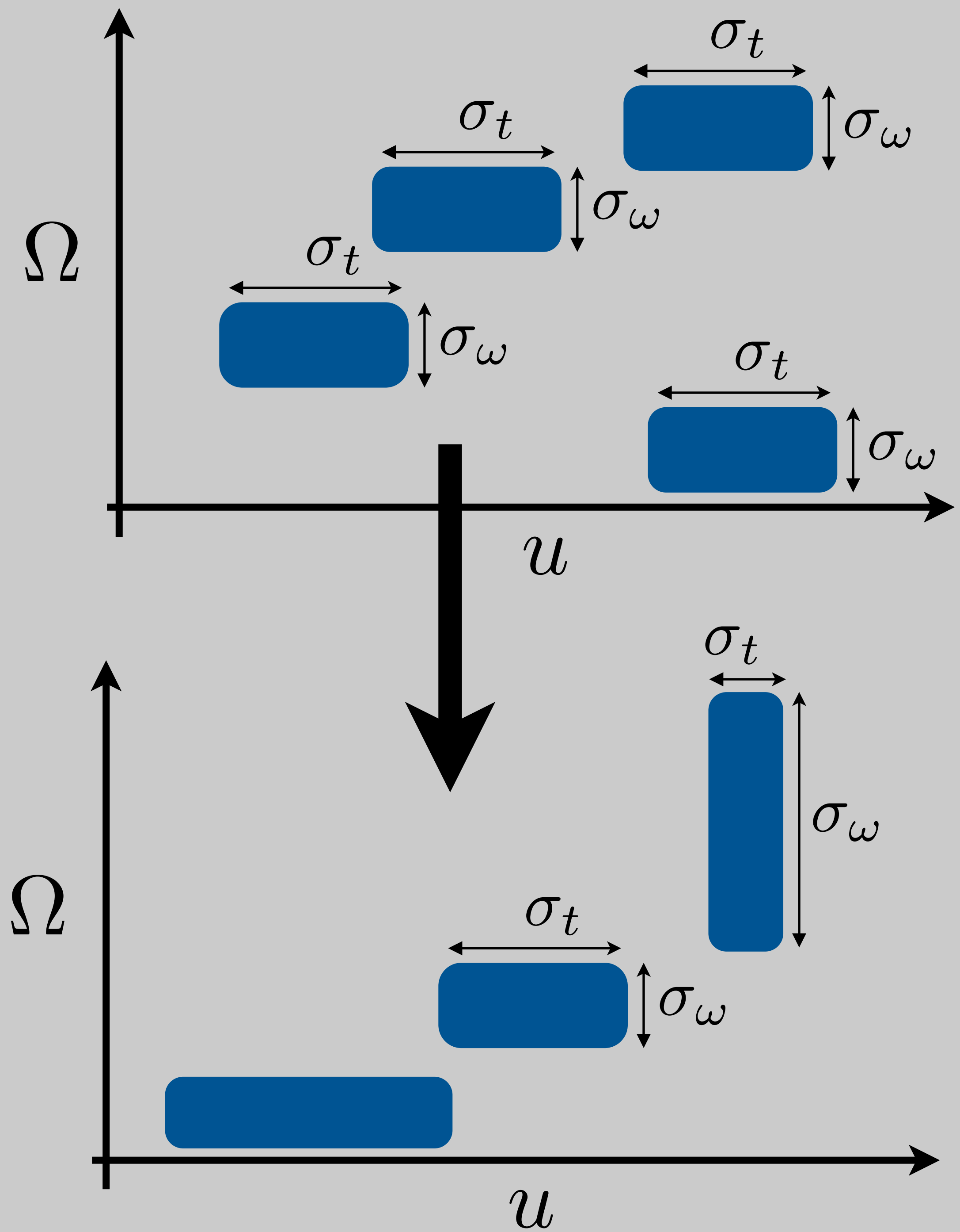
From STFT to Wavelets

- Basic Idea:
  - –low-freq changes slowly - fast tracking unimportant
  - –Fast tracking of high-freq is important in many apps.
  - –Must adapt Heisenberg box to frequency

- Back to continuous time for a bit.....

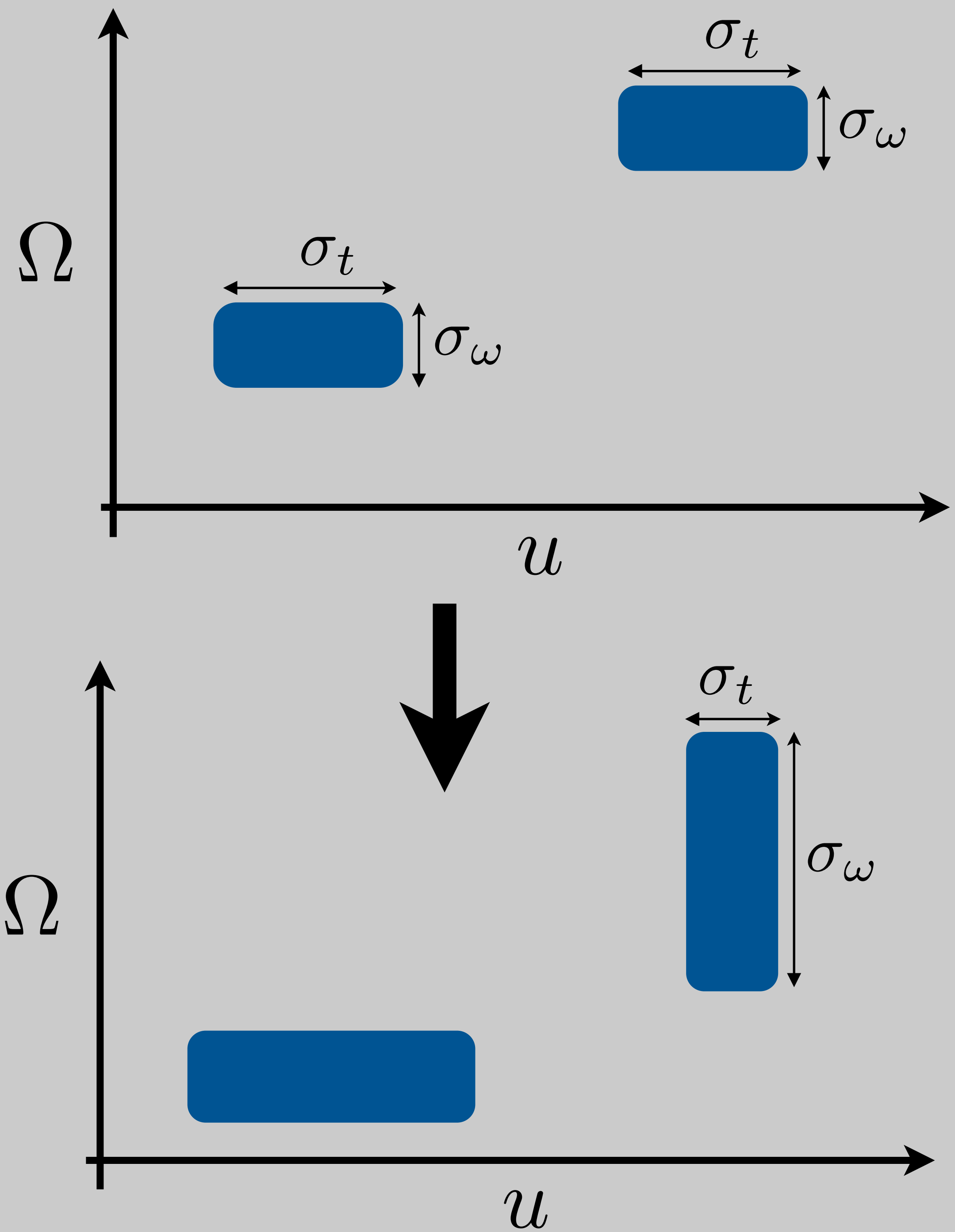
From STFT to Wavelets

- Continuous time



From STFT to Wavelets

- Continuous time



$$Sf(u,\Omega) = \int_{-\infty}^{\infty} f(t)w(t-u)e^{-j\Omega t}dt$$

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\Psi^*(\frac{t-u}{s})dt$$

*Morlet - Grossmann

From STFT to Wavelets

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\Psi^*(\frac{t-u}{s})dt$$

- The function Ψ is called a mother wavelet

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1$$ $\Rightarrow$ unit norm

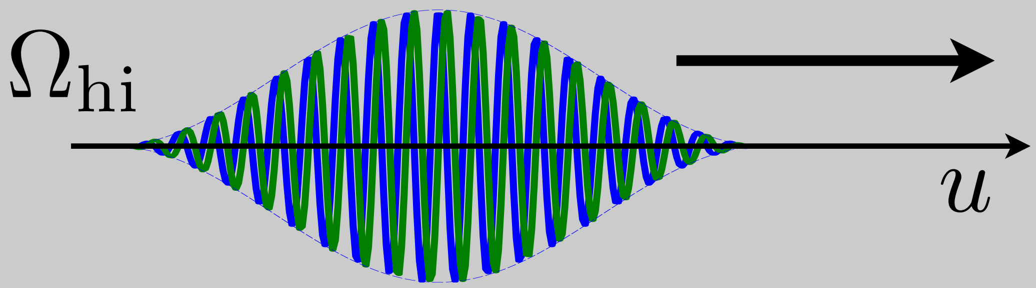
$$\int_{-\infty}^{\infty} \Psi(t) dt = 0$$ $\Rightarrow$ Band-Pass

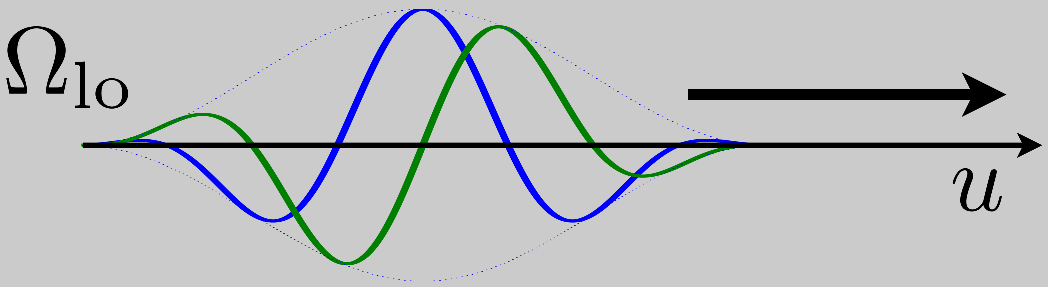
STFT and Wavelets “Atoms”

STFT Atoms

(with hamming window)

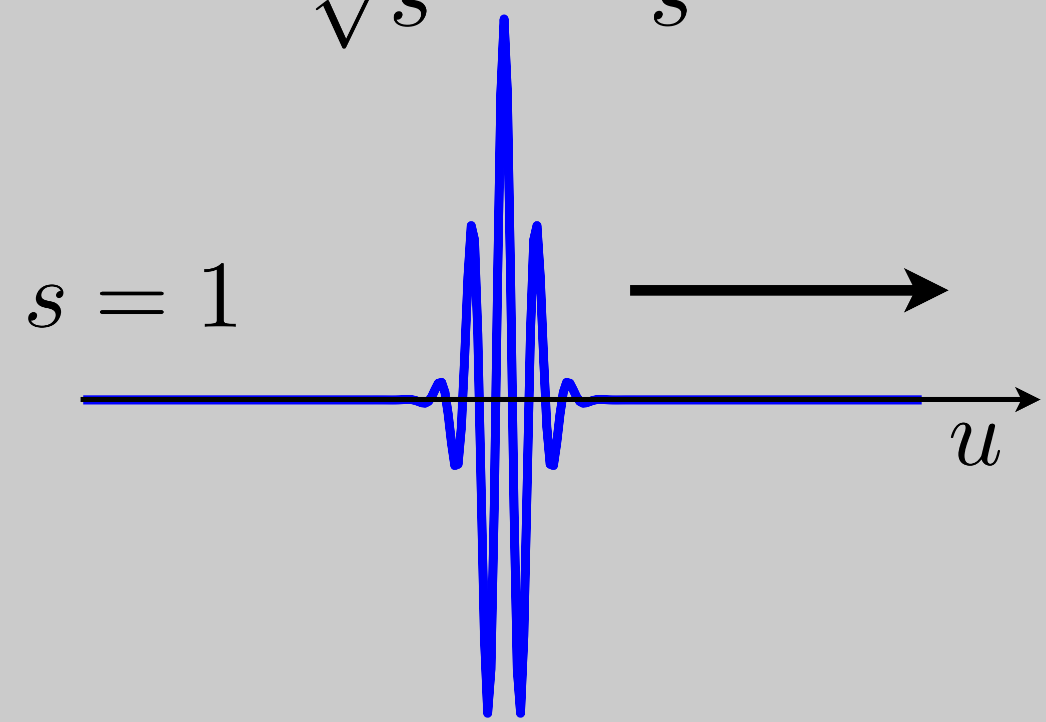
$$w(t-u)e^{j\Omega t}$$

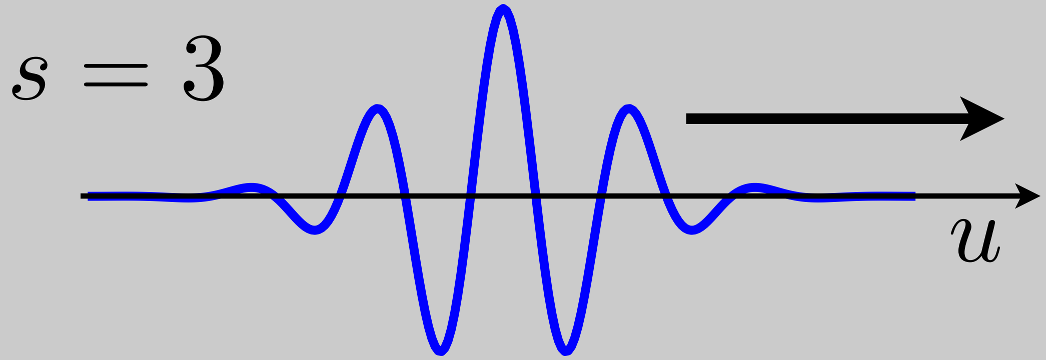




Wavelet Atoms

$$\frac{1}{\sqrt{s}}\Psi\left(\frac{t-u}{s}\right)$$

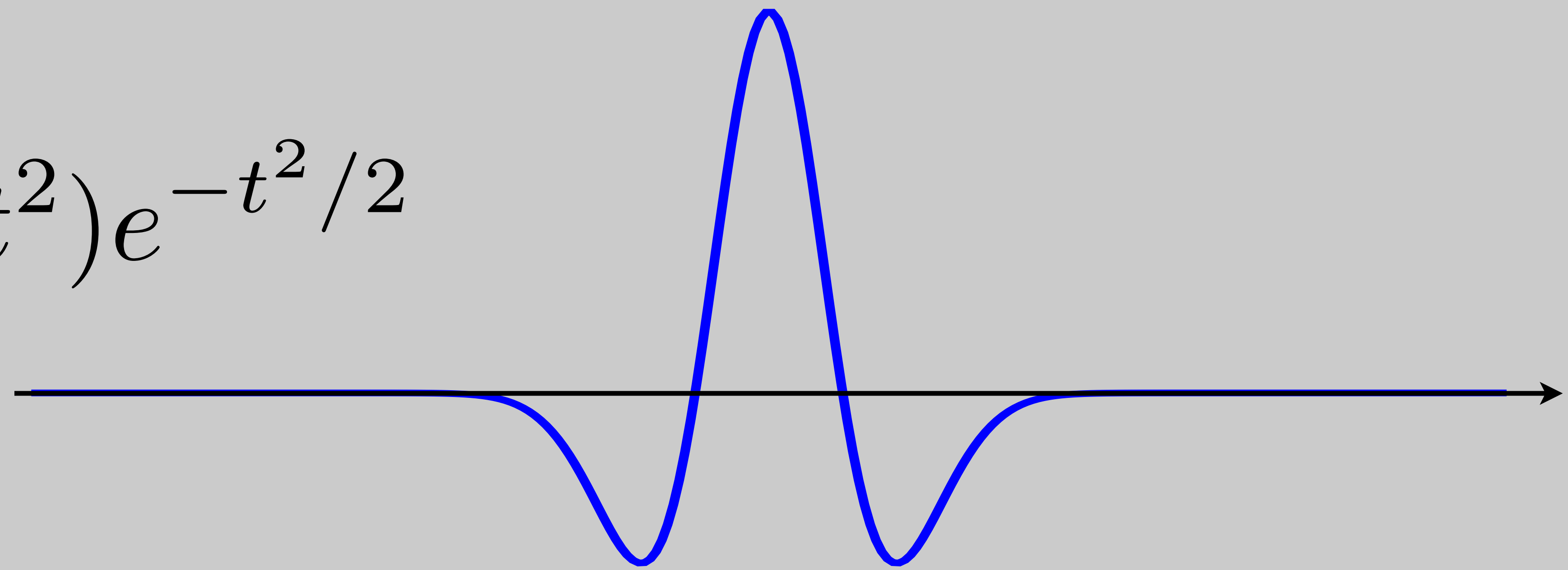




Examples of Wavelets

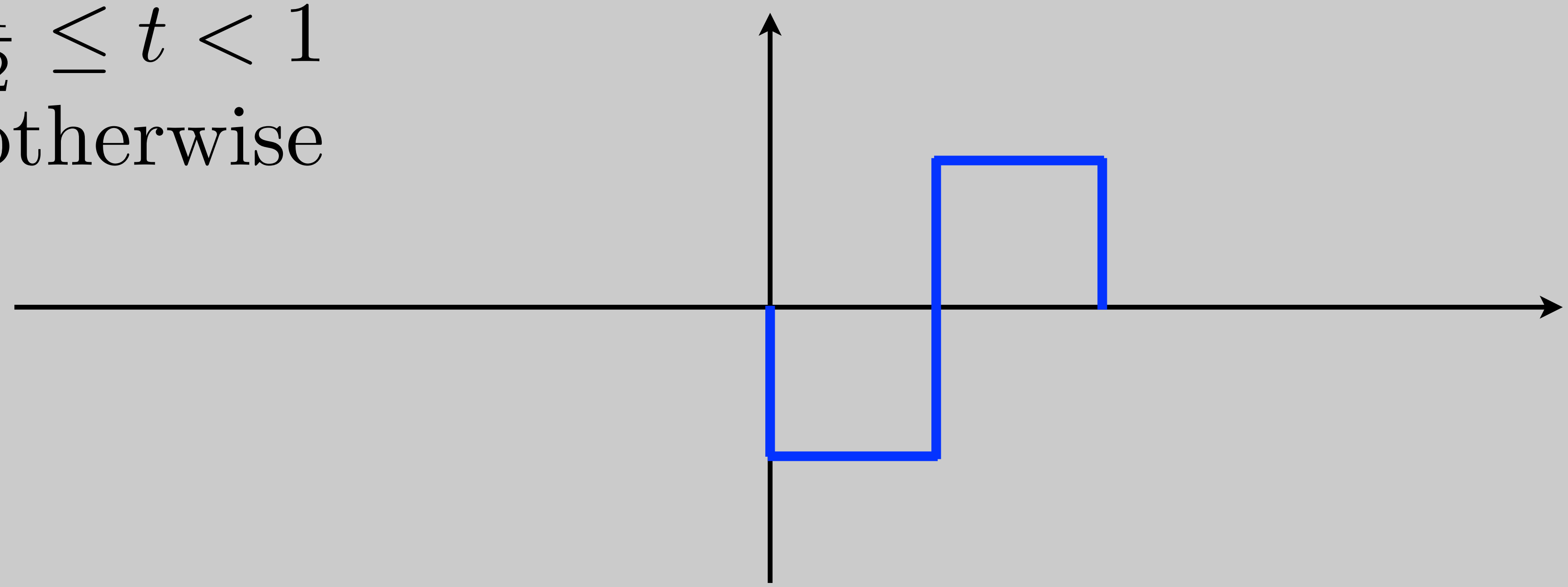
- Mexican Hat

$$\Psi(t) = (1-t^2)e^{-t^2/2}$$



- Haar

$$\Psi(t) = \begin{cases} -1 & 0 \leq t < \frac{1}{2} \\ 1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



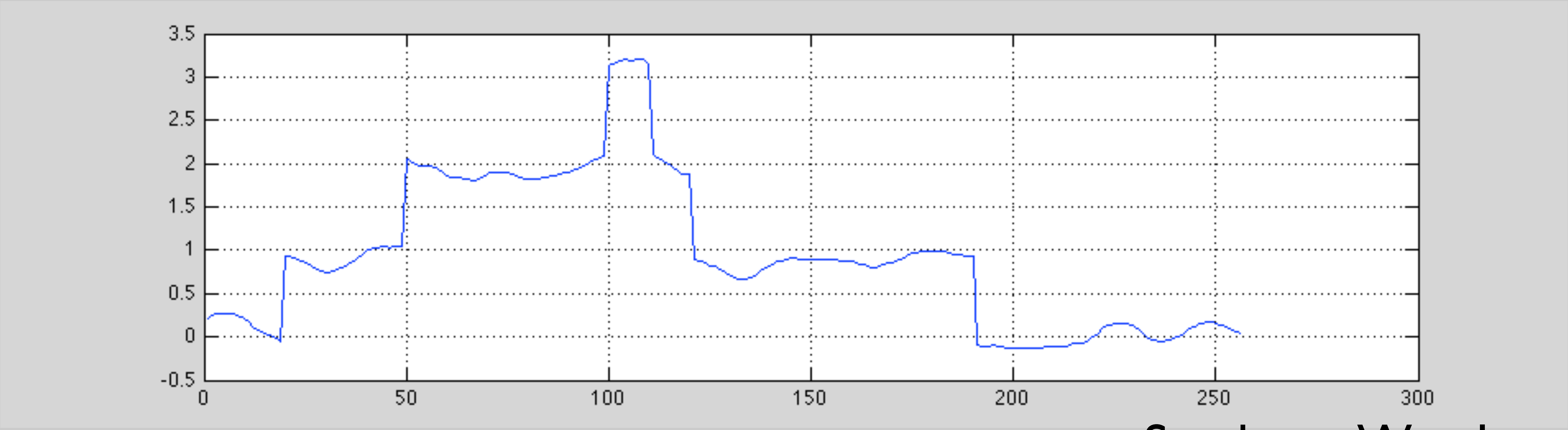
- Can be written as linear filtering

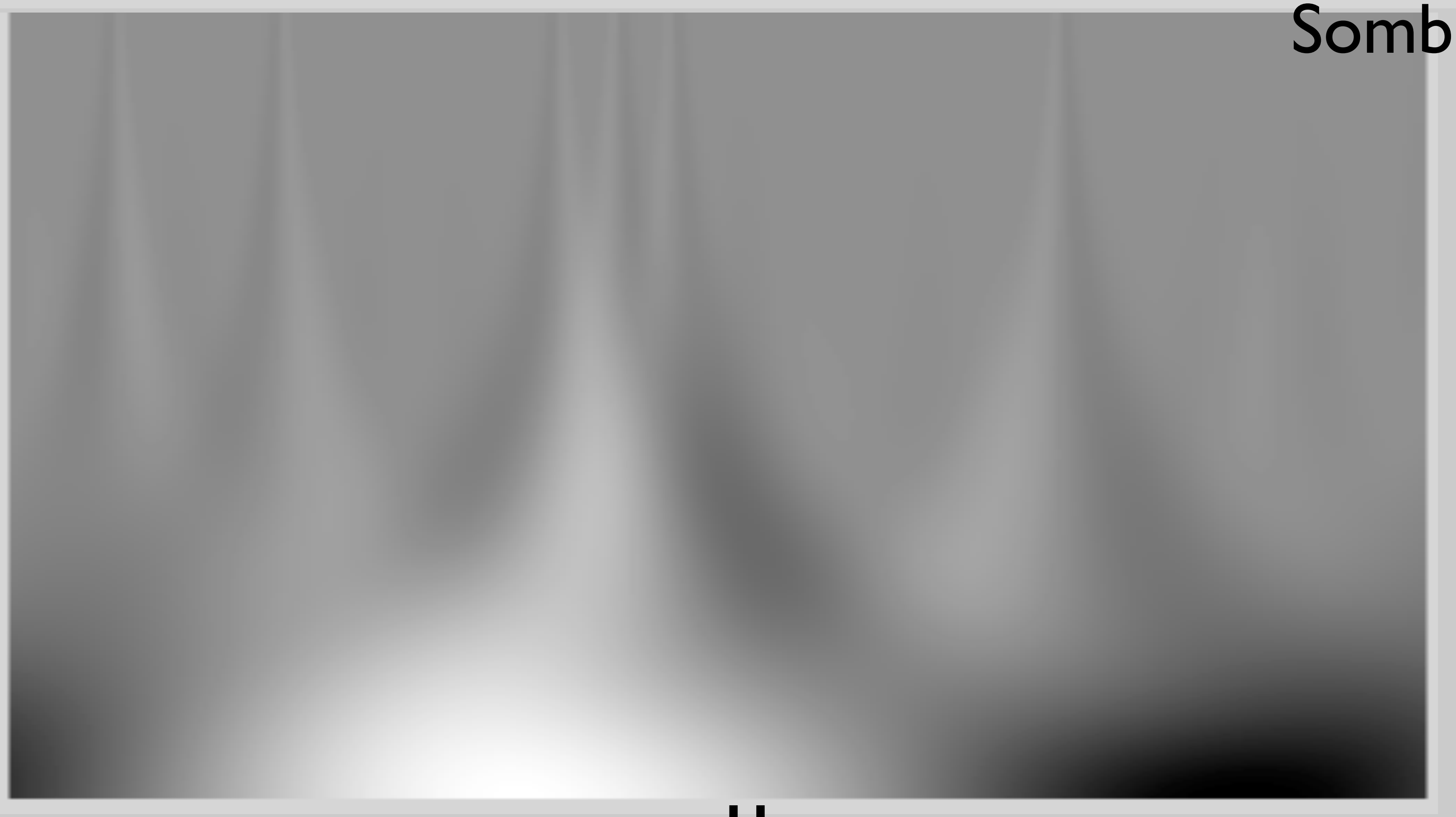
$$
\begin{aligned}
Wf(u,s) &= \frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} f(t)\Psi^*(\frac{t-u}{s})dt \\
&= \left\{f(t) * \overline{\Psi}_s(t)\right\}(u)
\end{aligned}
$$

$$\overline{\Psi}_s = \frac{1}{\sqrt{s}}\Psi(\frac{t}{s})$$

- Wavelet coefficients are a result of bandpass filtering

Example 2: "Bumpy" Signal





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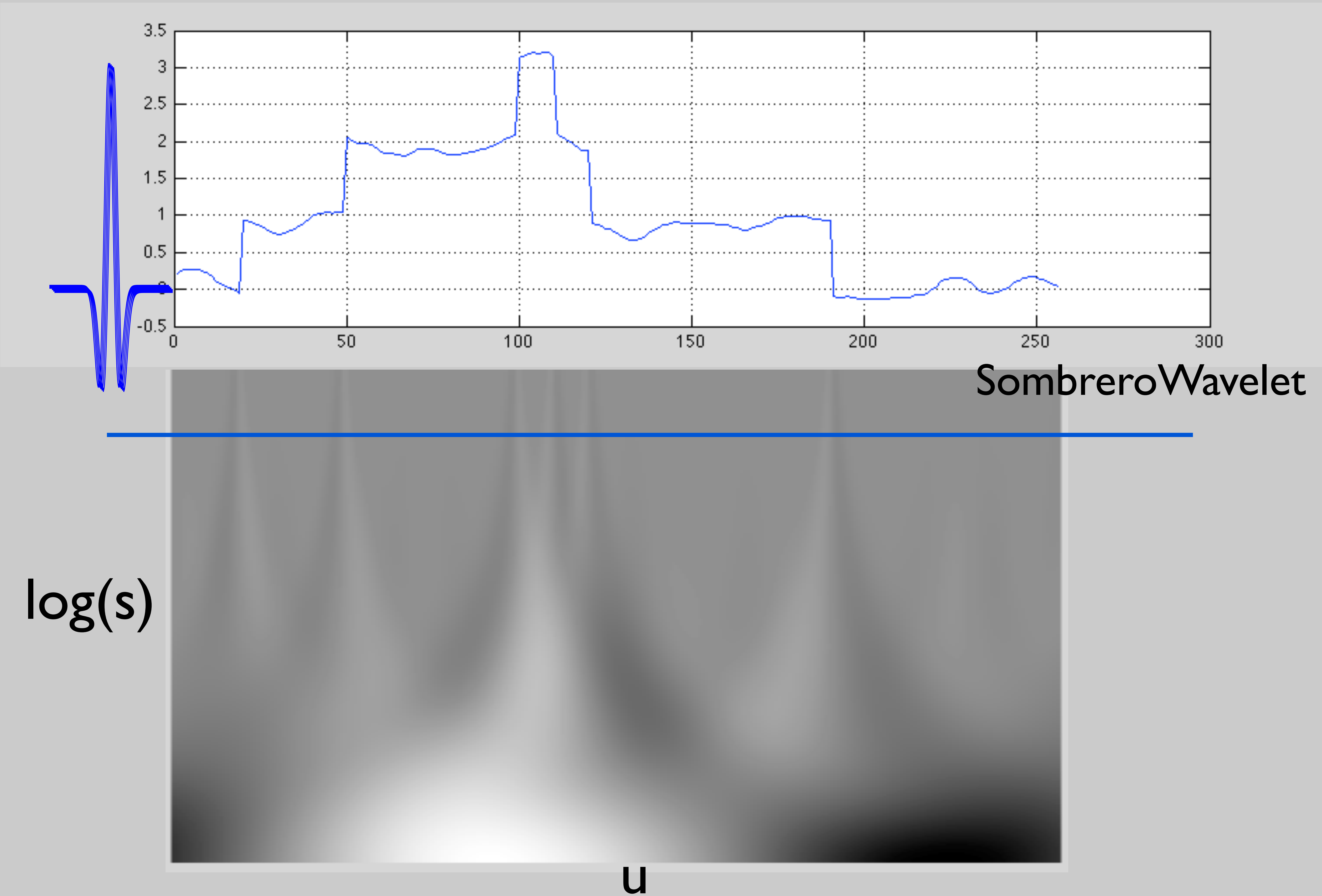

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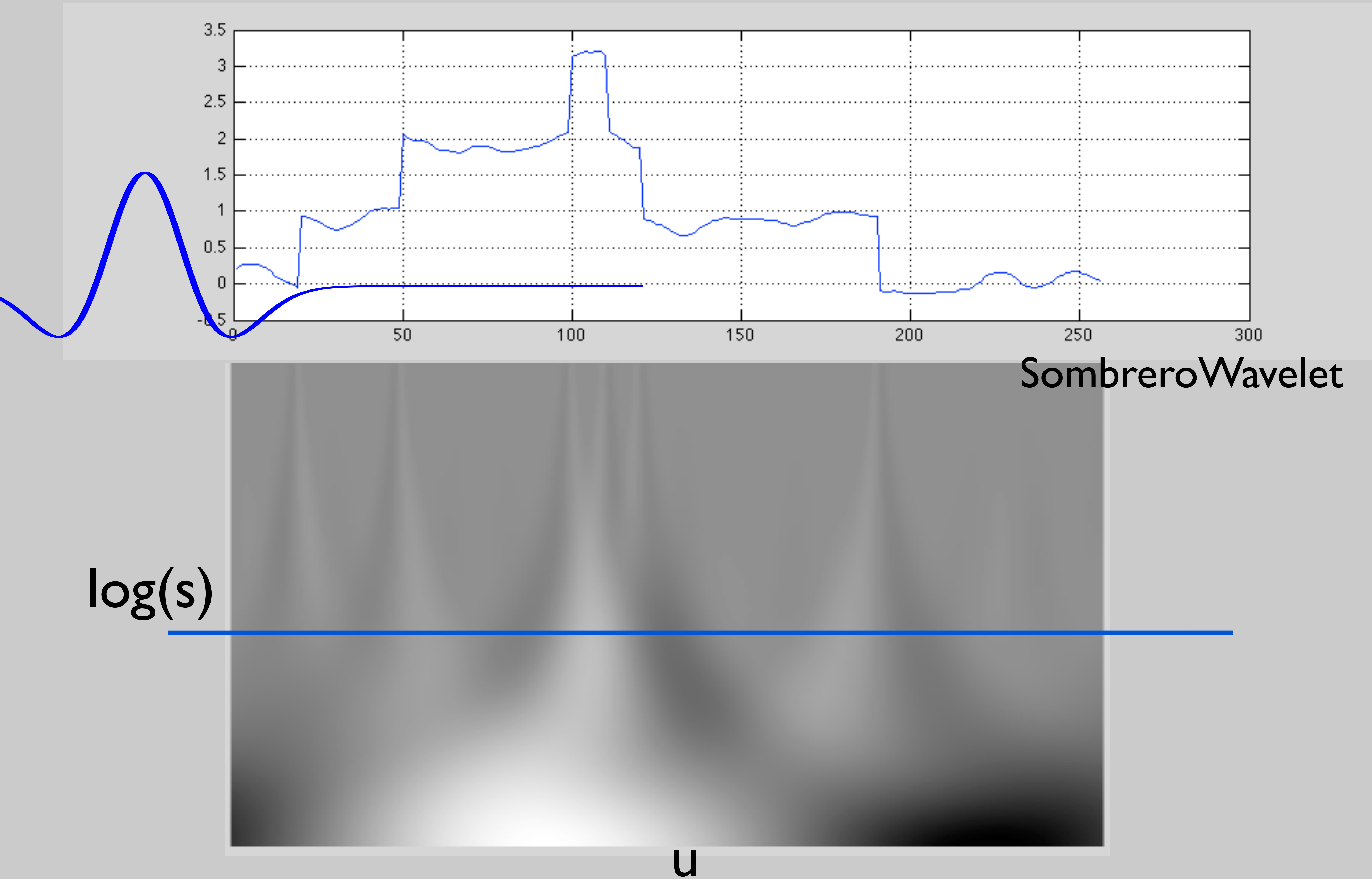

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