EE123
Digital Signal Processing

Lecture 6A
Wavelets Cont.

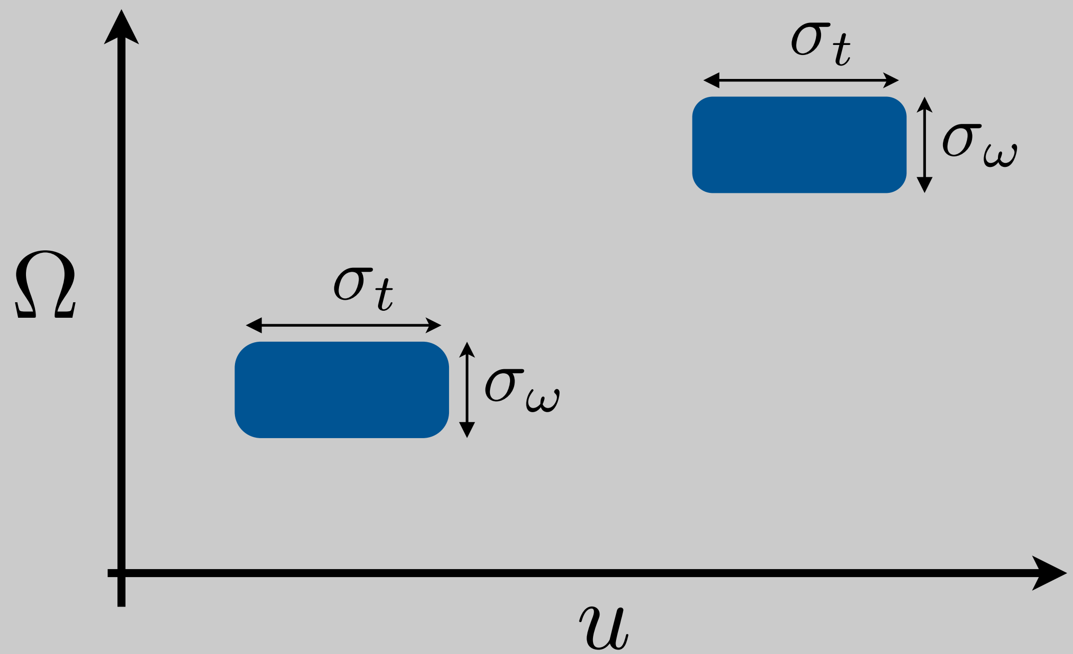
M. Lustig, EECS UC Berkeley

Announcements

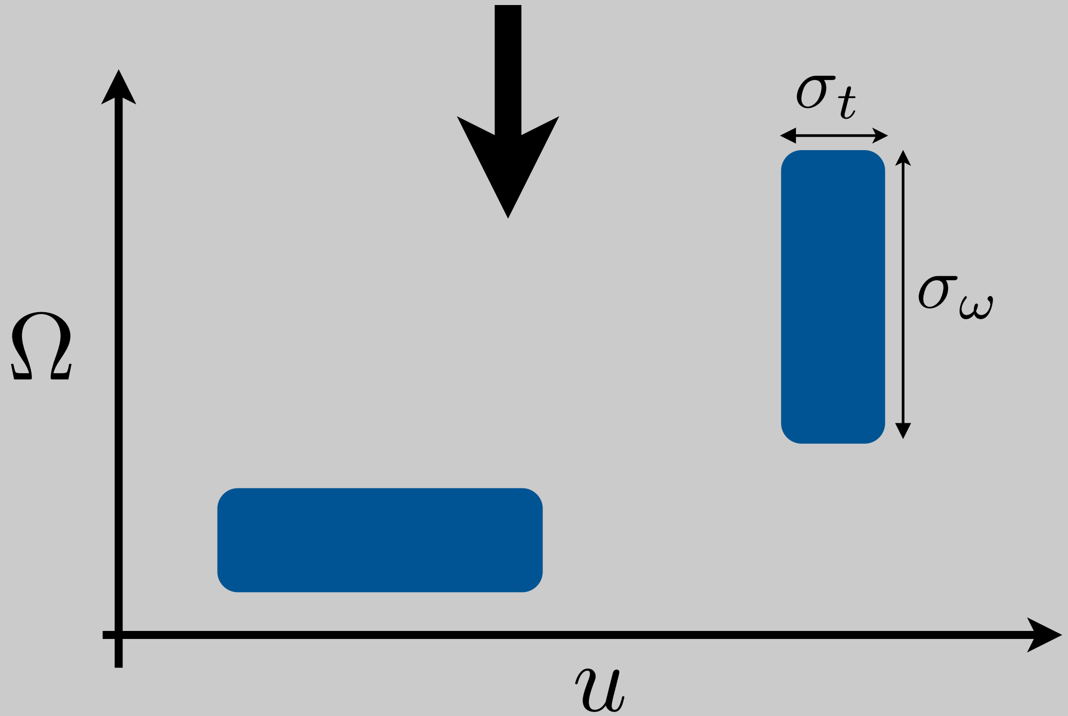
- Midterm I
  - Wednesday 2-4pm in class
  - Everything up to Wavelets
  - Open everything -- except electronics
- Pre-Lab II + Lab II Part I
  - Install SDR drivers + Software on laptops
  - Look at different parts of the spectrum
  - Look at the effect of windowing on spectrum

From STFT to Wavelets

- Continuous time



$$Sf(u,\Omega) = \int_{-\infty}^{\infty} f(t)w(t-u)e^{-j\Omega t}dt$$



$$Wf(u,s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\Psi^*(\frac{t-u}{s})dt$$

*Morlet - Grossmann

From STFT to Wavelets

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*(\frac{t-u}{s}) dt$$

- The function Ψ is called a mother wavelet
  - –Must satisfy:

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1$$ ⇒ unit norm

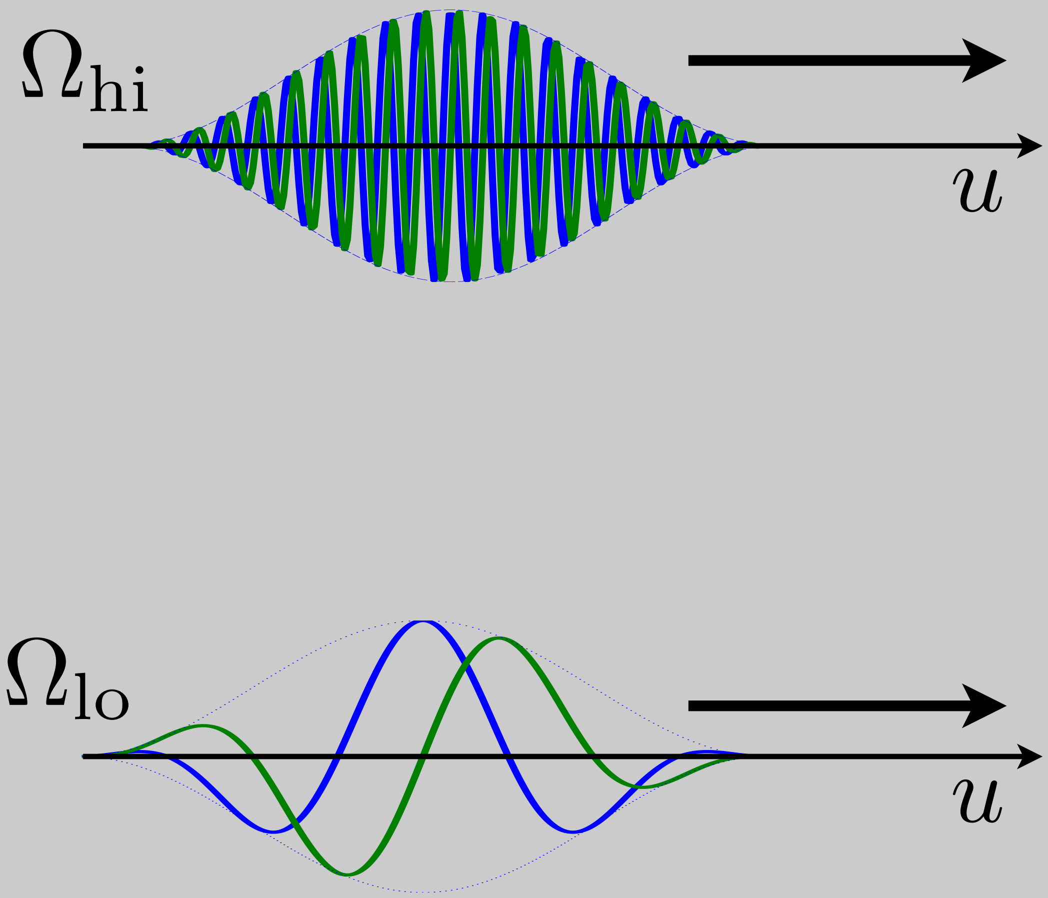
$$\int_{-\infty}^{\infty} \Psi(t) dt = 0$$ ⇒ Band-Pass

STFT and Wavelets "Atoms"

STFT Atoms

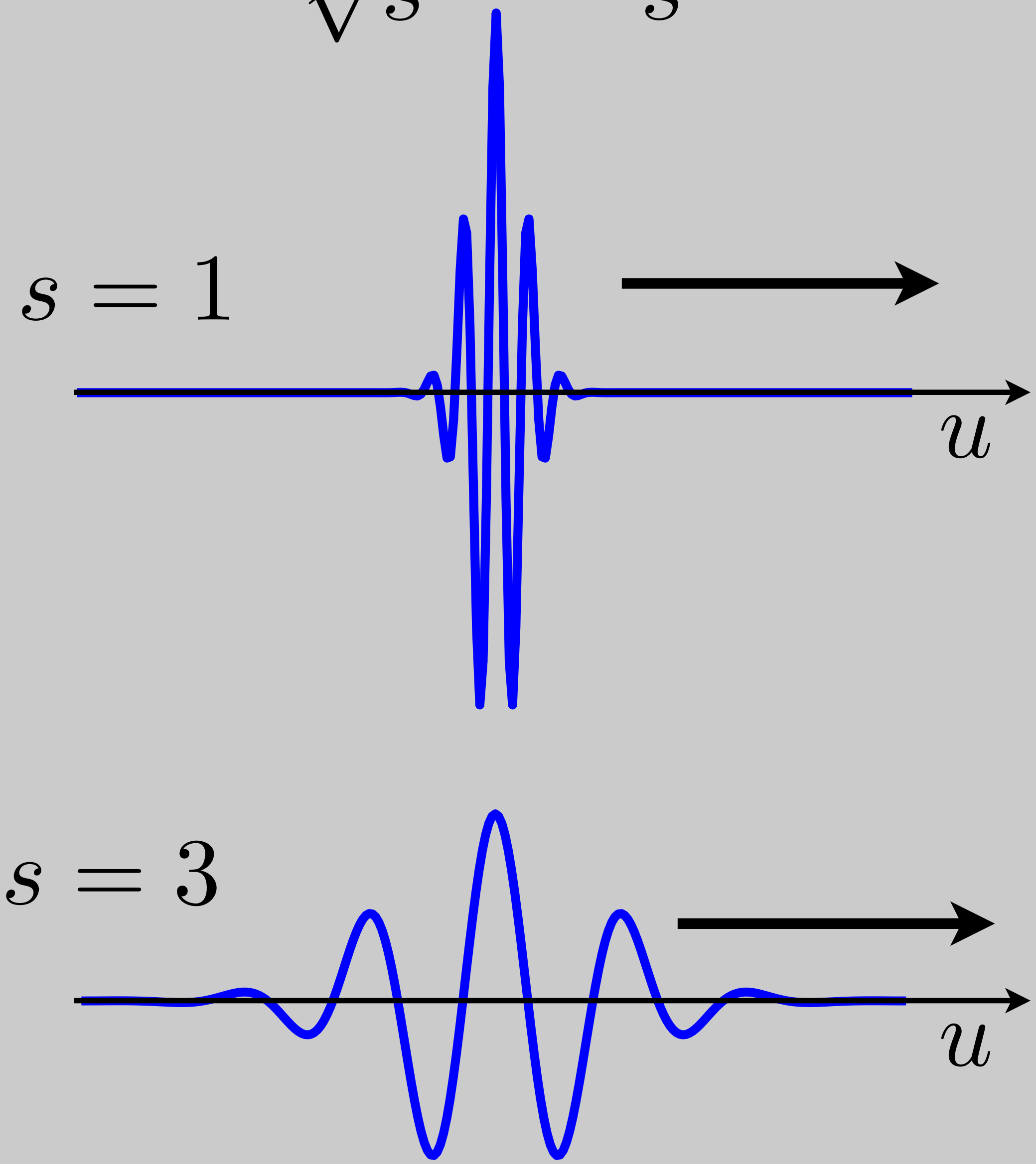
(with hamming window)

$$w(t-u)e^{j\Omega t}$$



Wavelet Atoms

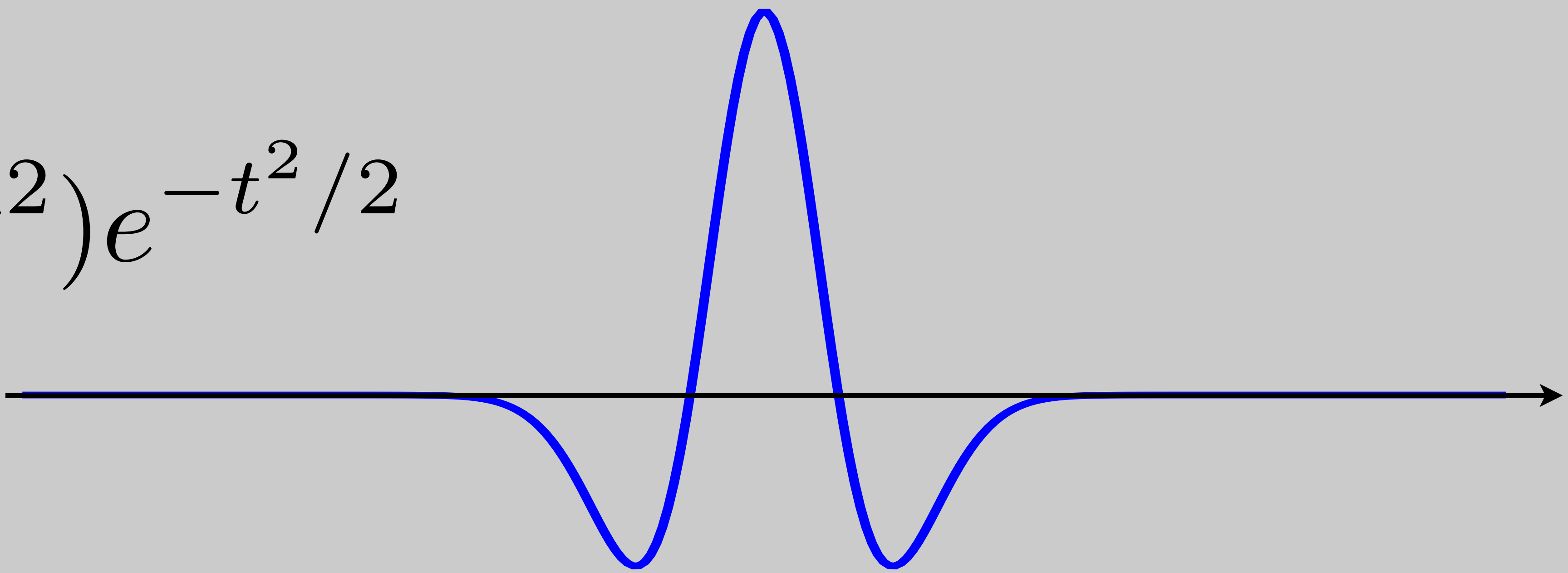
$$\frac{1}{\sqrt{s}}\Psi\left(\frac{t-u}{s}\right)$$



Examples of Wavelets

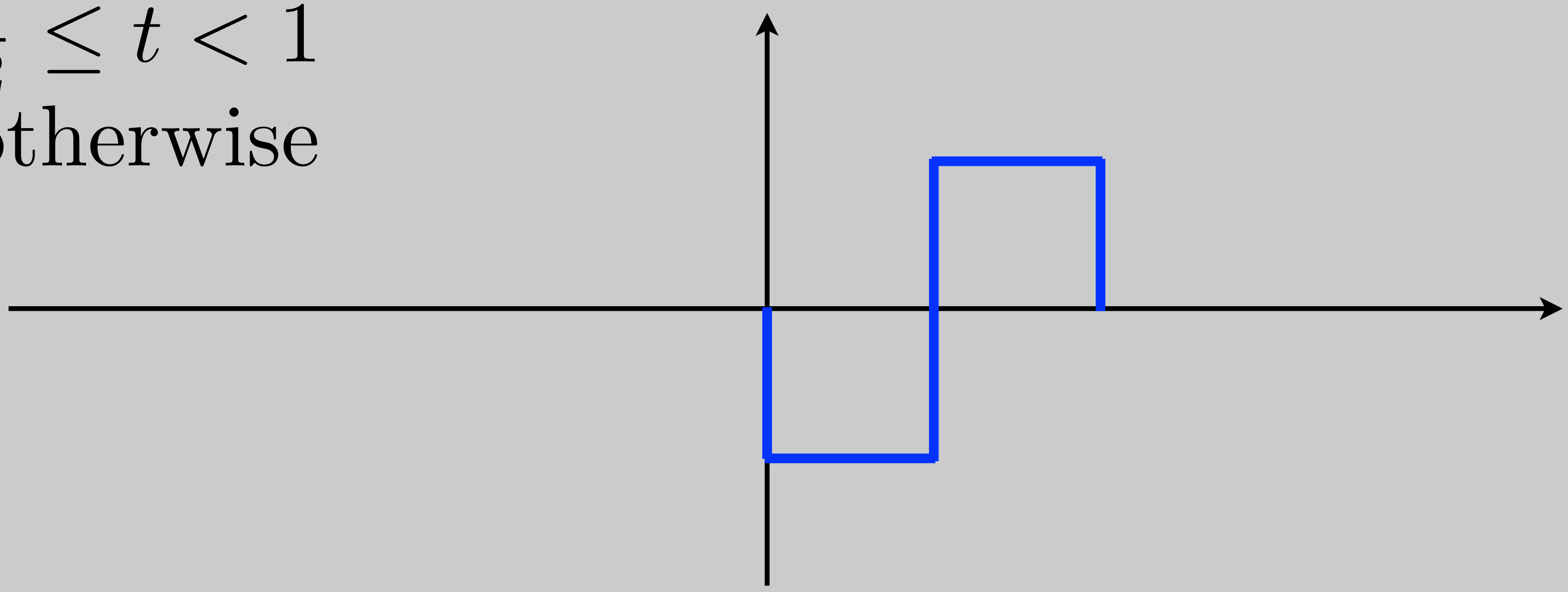
- Mexican Hat

$$\Psi(t) = (1 - t^2)e^{-t^2/2}$$



- Haar

$$\Psi(t) = \begin{cases} -1 & 0 \leq t < \frac{1}{2} \\ 1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



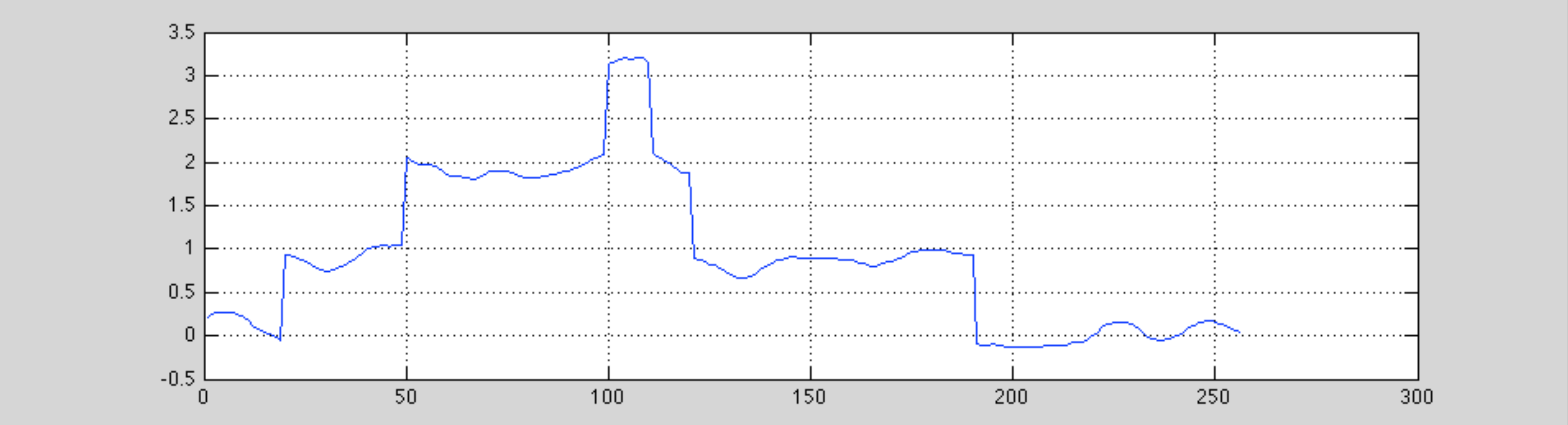
- Can be written as linear filtering

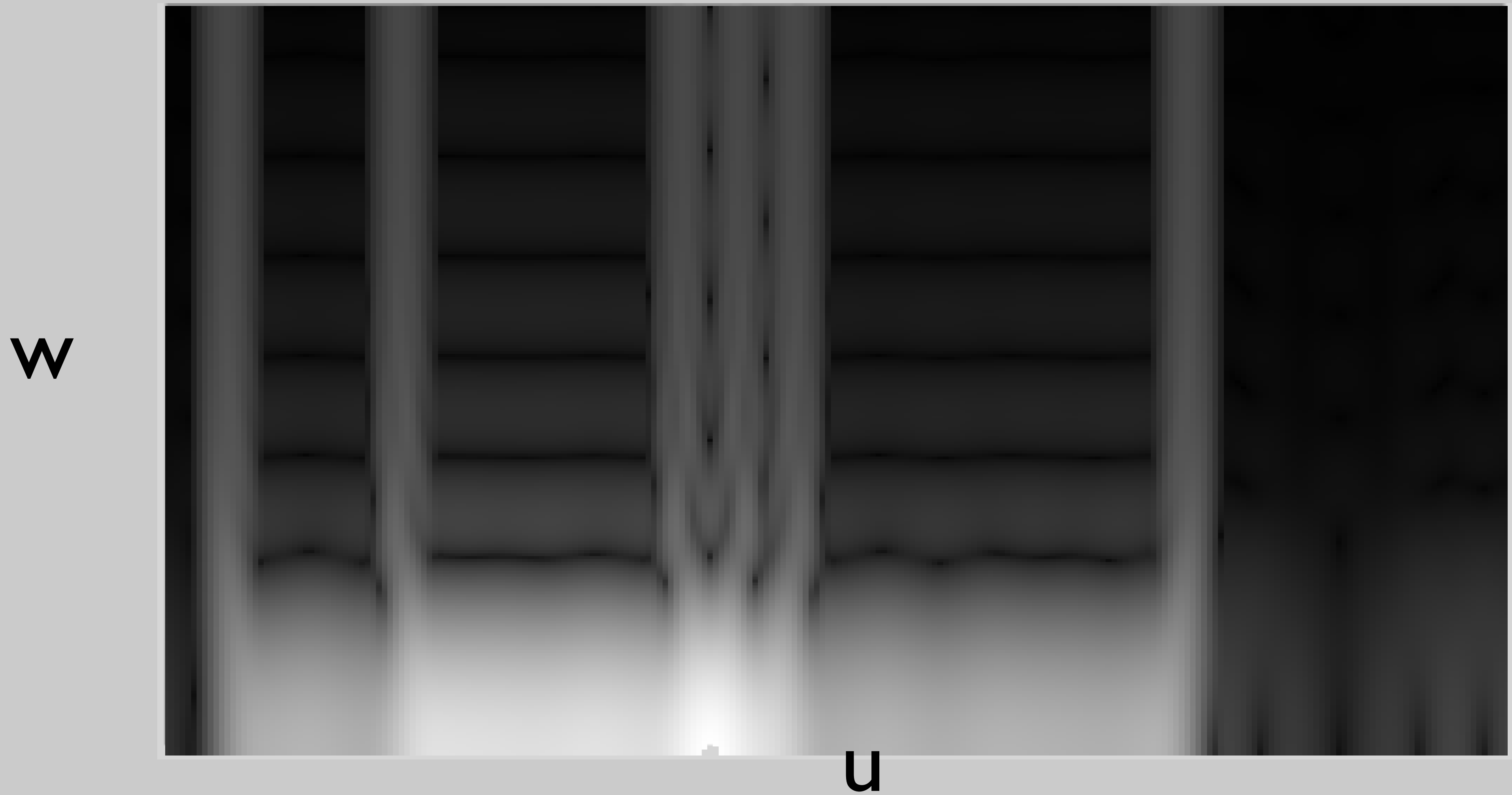
$$
\begin{aligned}
Wf(u,s) &= \frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} f(t)\Psi^*(\frac{t-u}{s})dt \\
&= \left\{f(t) * \overline{\Psi}_s(t)\right\}(u)
\end{aligned}
$$

$$\overline{\Psi}_s = \frac{1}{\sqrt{s}}\Psi(\frac{t}{s})$$

- Wavelet coefficients are a result of bandpass filtering

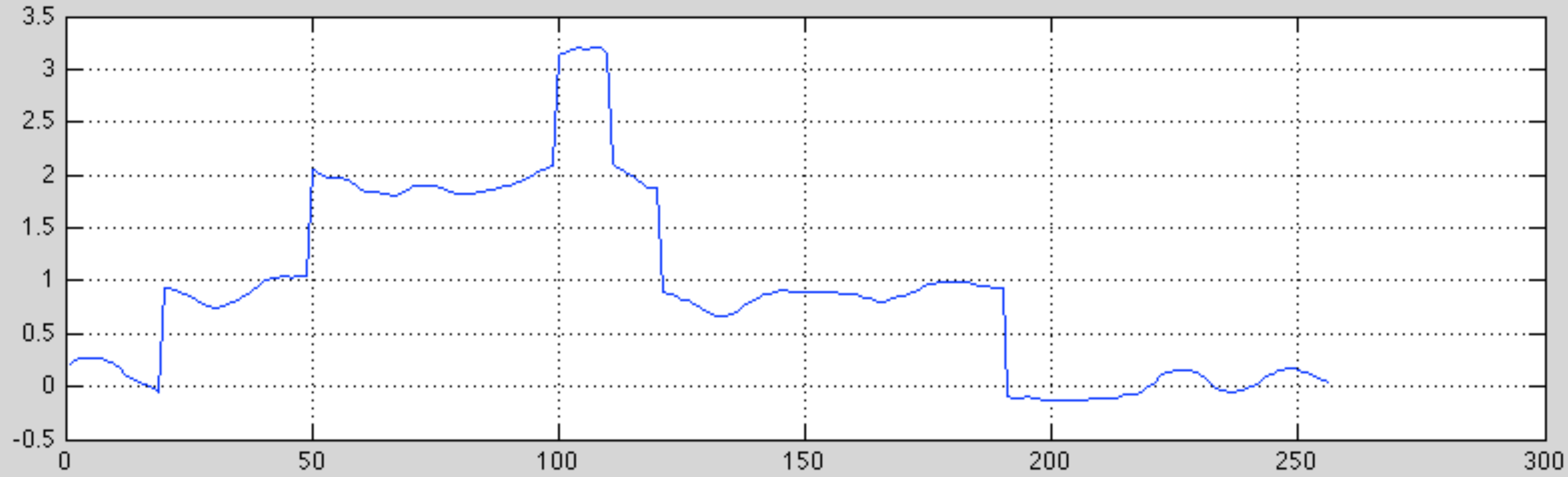
Example 2: “Bumpy” Signal



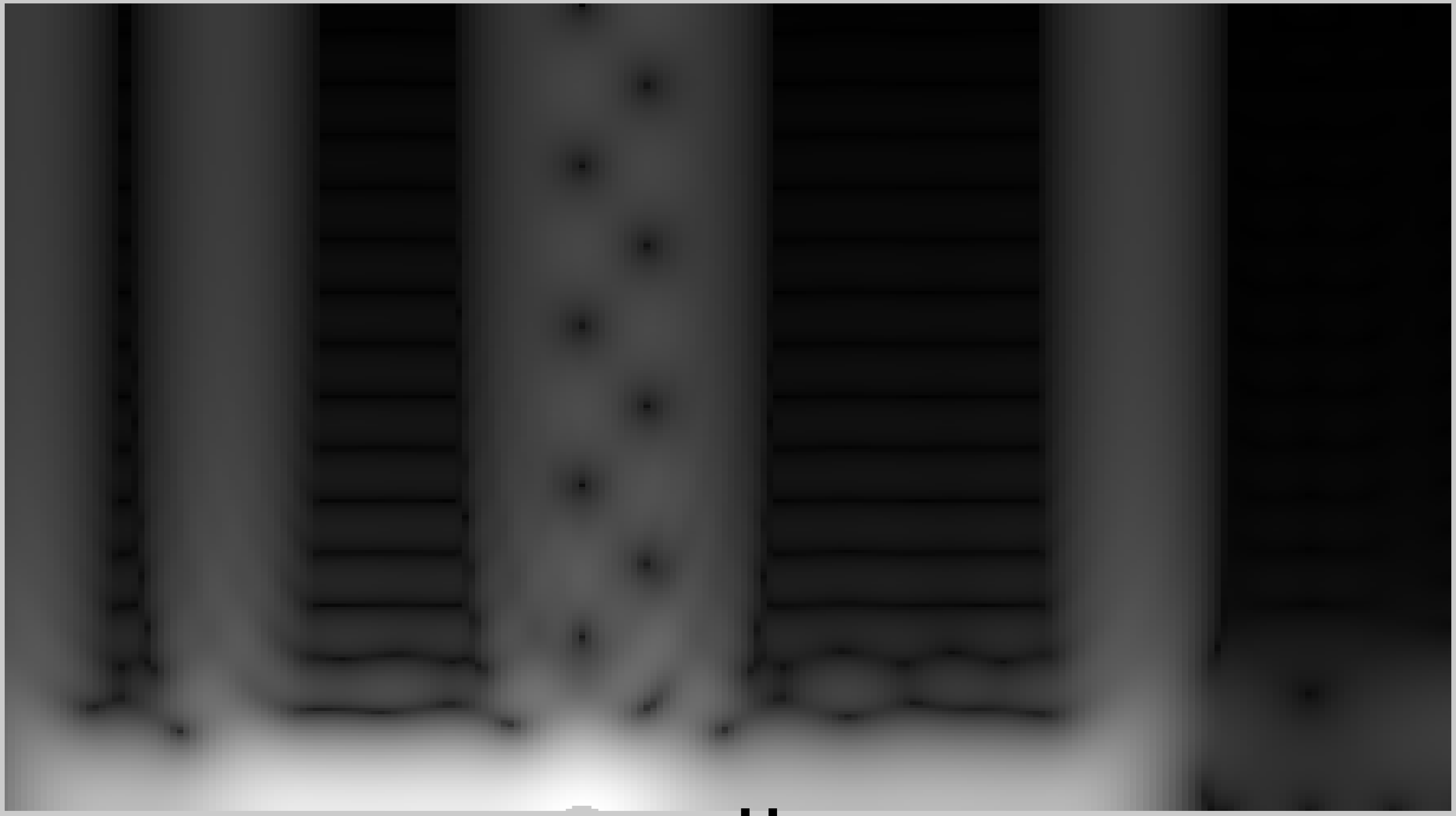


STFT
L = 16
R = 1
N = 512
Hann

Example 2: “Bumpy” Signal



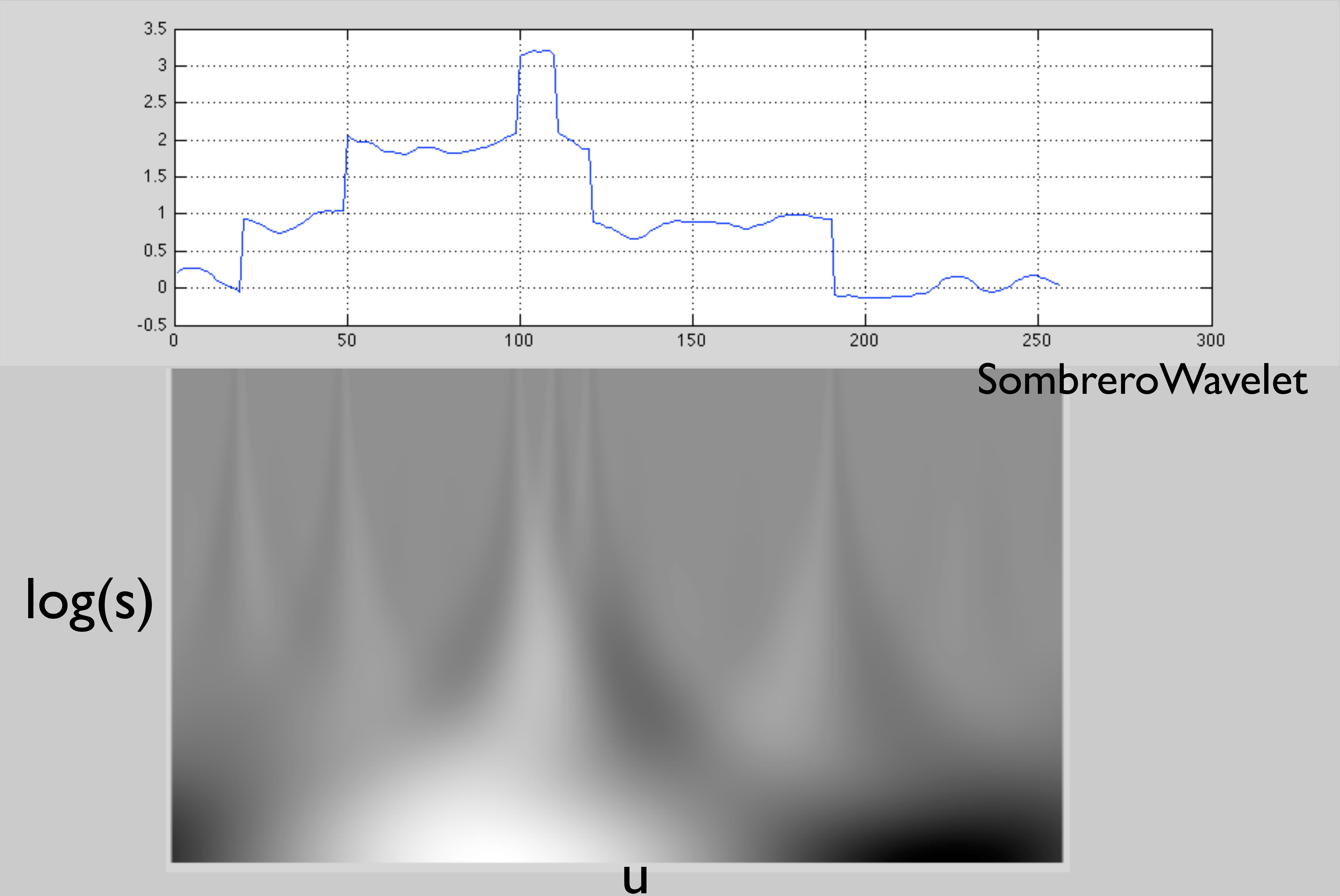
w



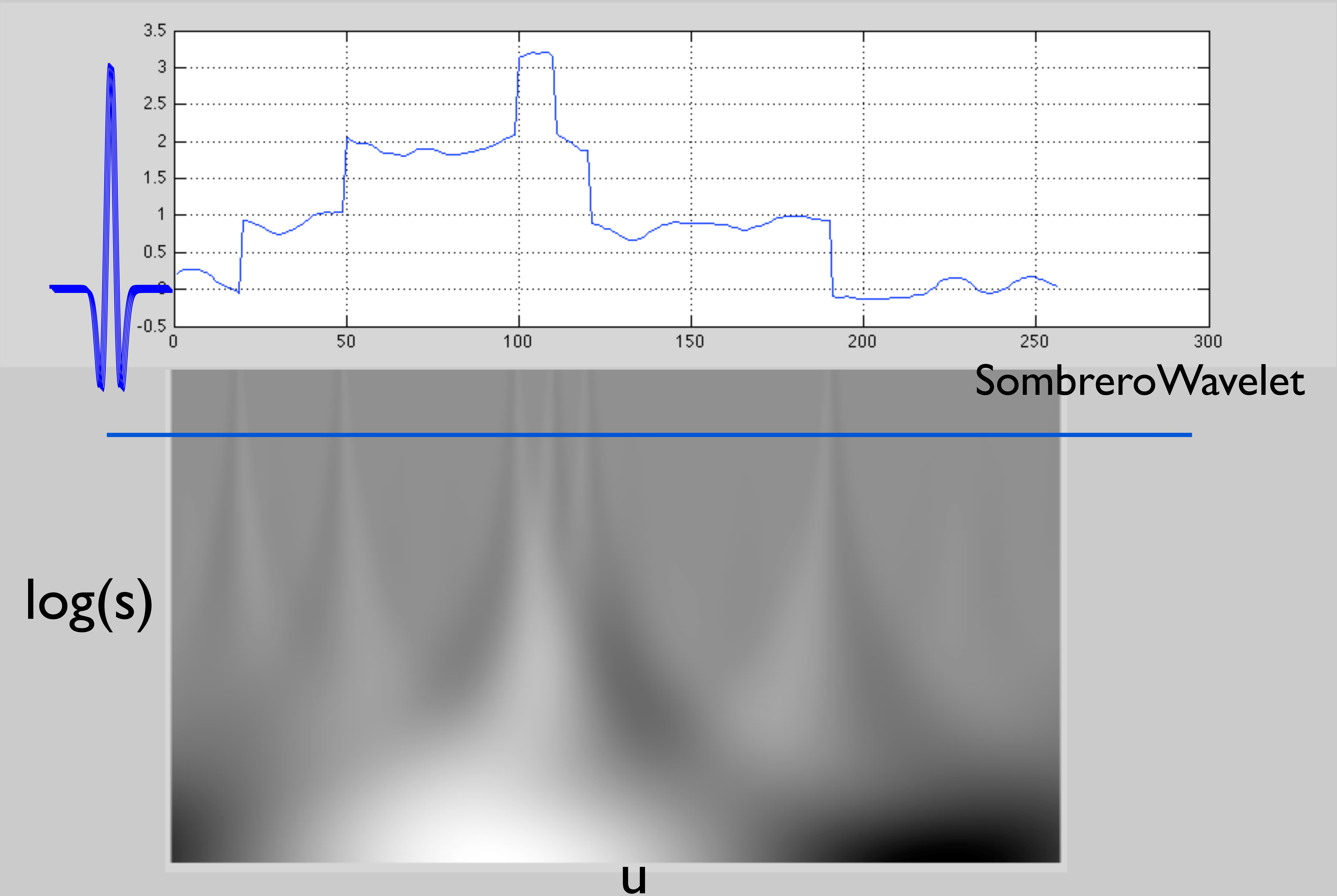
u

STFT
L = 32
R = 1
N = 512
Hann

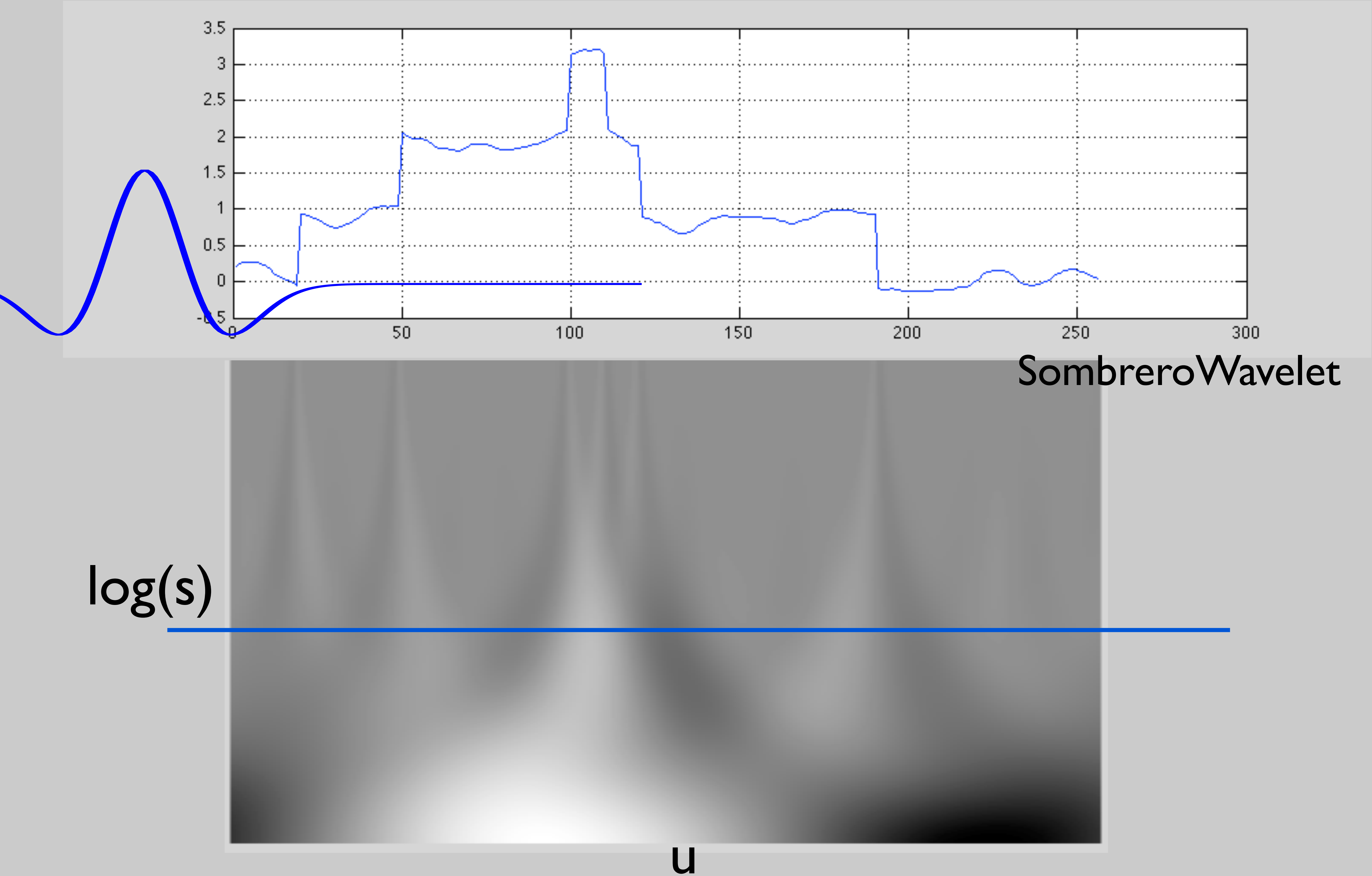
Example 2: “Bumpy” Signal



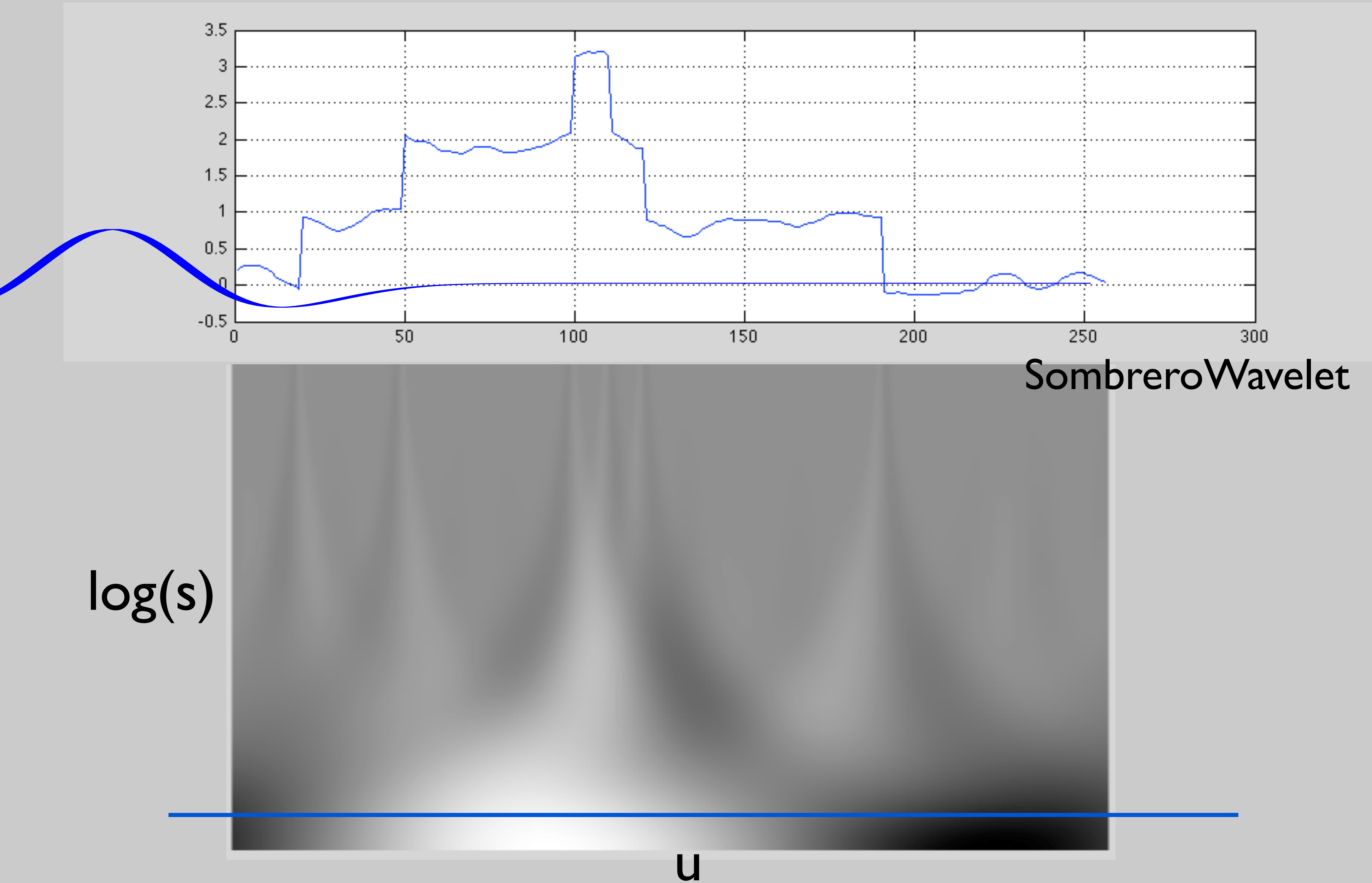
Example 2: "Bumpy" Signal



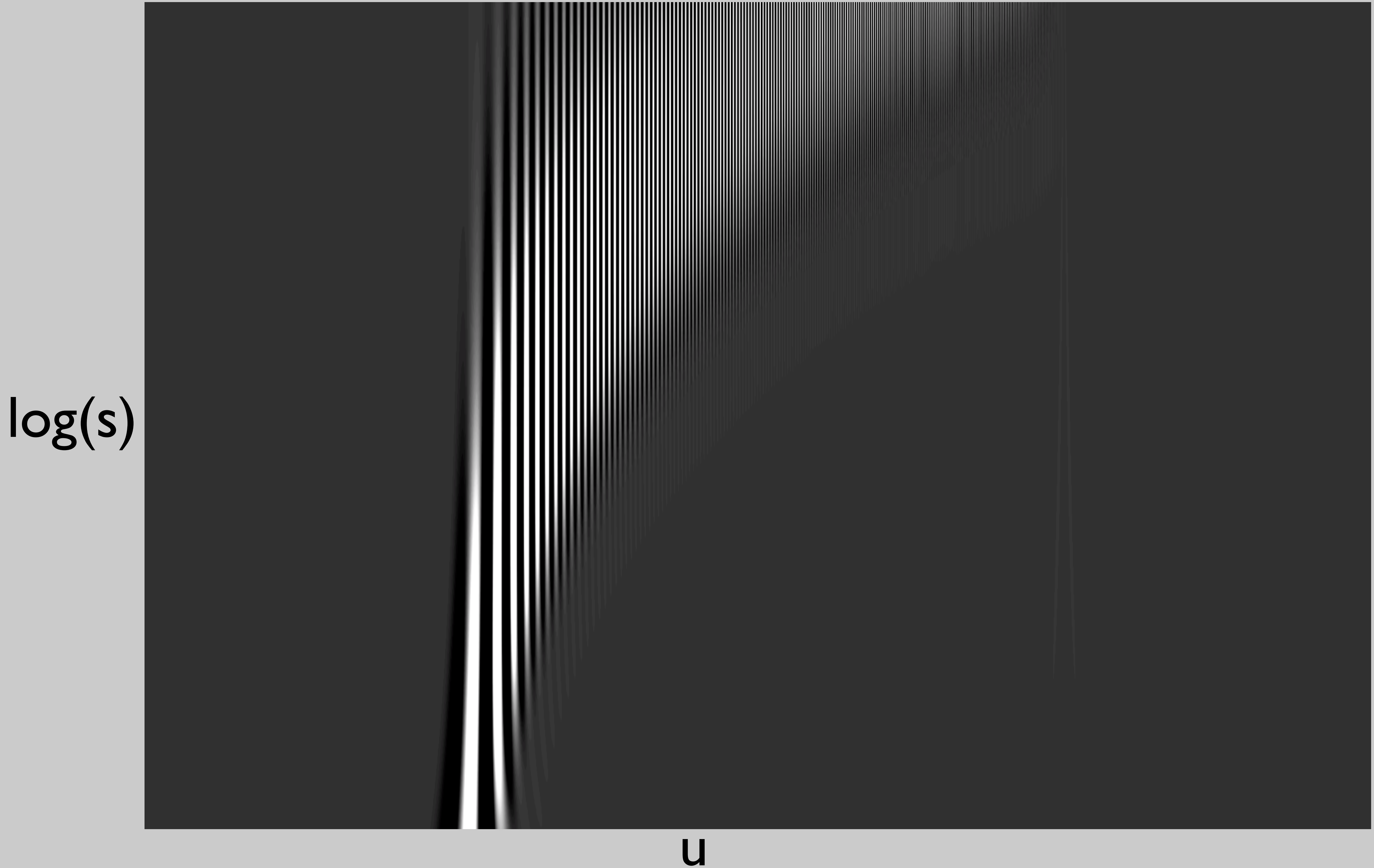
Example 2: “Bumpy” Signal



Example 2: “Bumpy” Signal



What's this function?



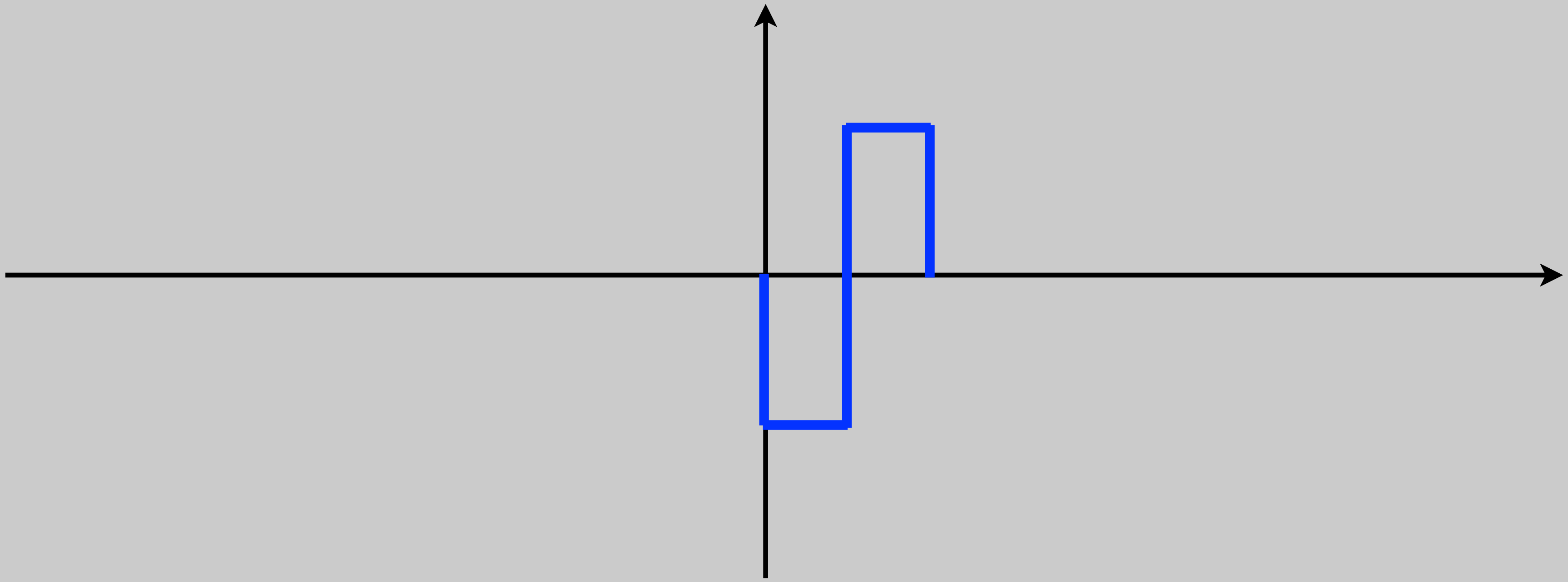
- Many different constructions for different signals
  - Haar good for piece-wise constant signals
  - Battle-Lemarie' : Spline polynomials

- Can construct Orthogonal wavelets
  - For example: dyadic Haar is orthonormal

$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$

$$i = [0, 1, 2, \cdots]$$

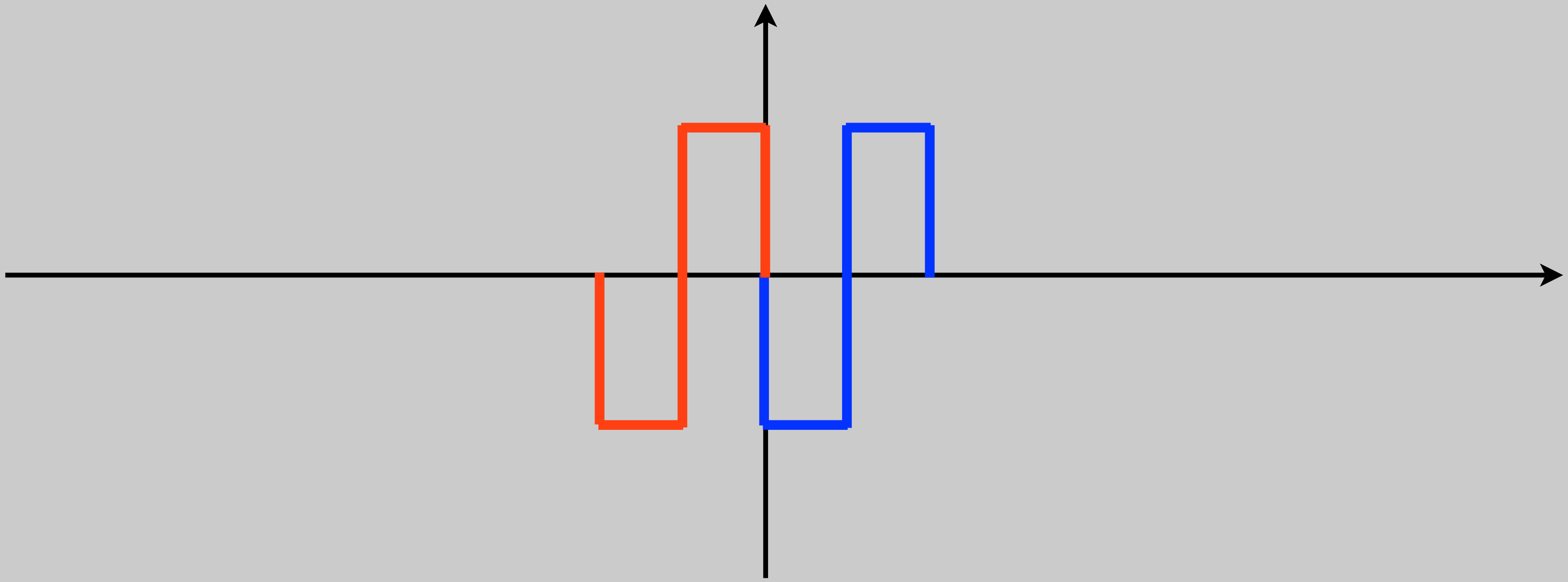
Orthonormal Haar - Basis functions



Same scale
non-overlapping

$$\overline{\Psi}_{0,0}(t) = \frac{1}{\sqrt{2^0}}\Psi(\frac{t-2^0 0}{2^0}) = \Psi(t)$$

Orthonormal Haar - Basis functions

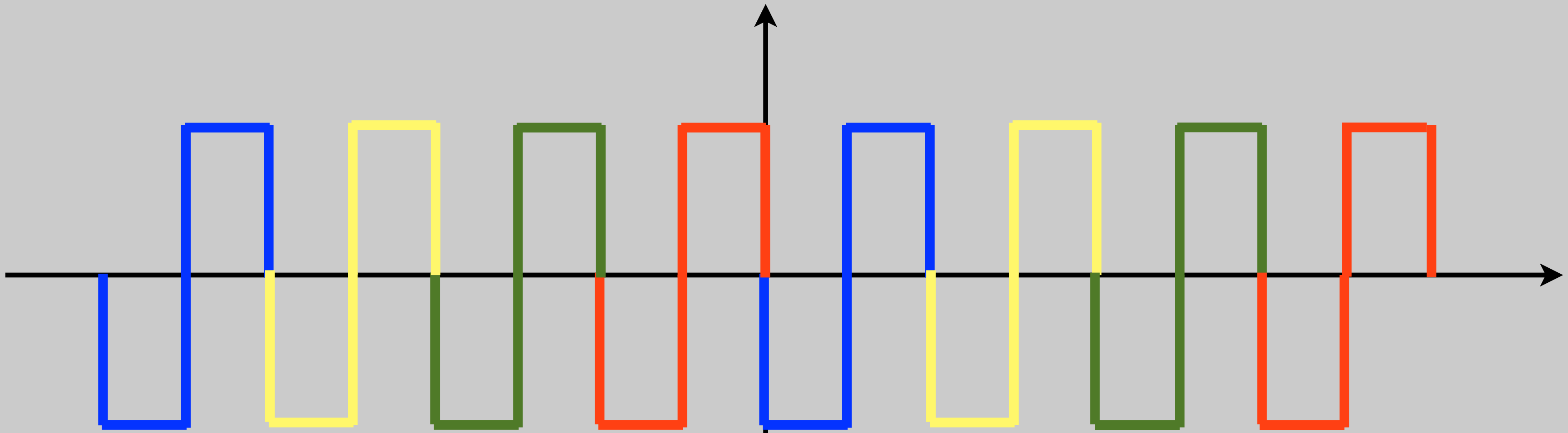


Same scale
non-overlapping

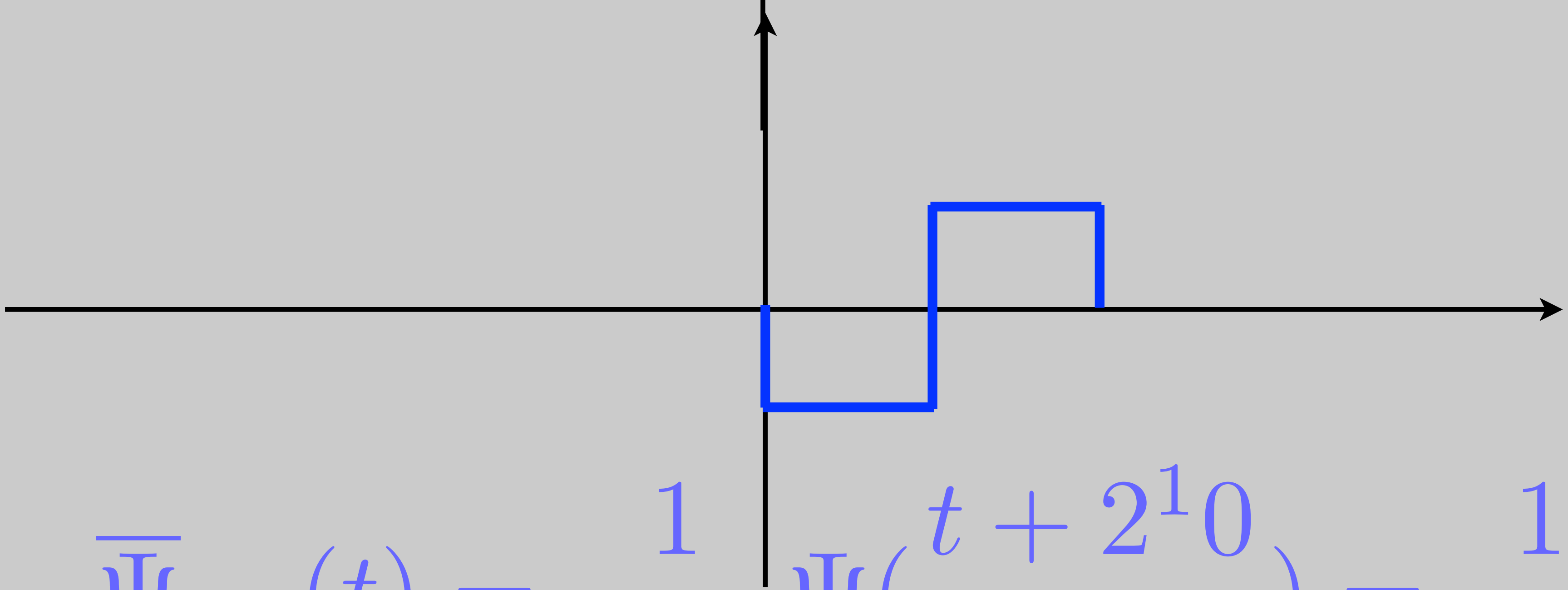
$$\overline{\Psi}_{0,0}(t) = \frac{1}{\sqrt{2^0}}\Psi(\frac{t - 2^0 0}{2^0}) = \Psi(t)$$

$$\overline{\Psi}_{0,-1}(t) = \frac{1}{\sqrt{2^0}}\Psi(\frac{t + 2^0 1}{2^0}) = \Psi(t+1)$$

Orthonormal Haar



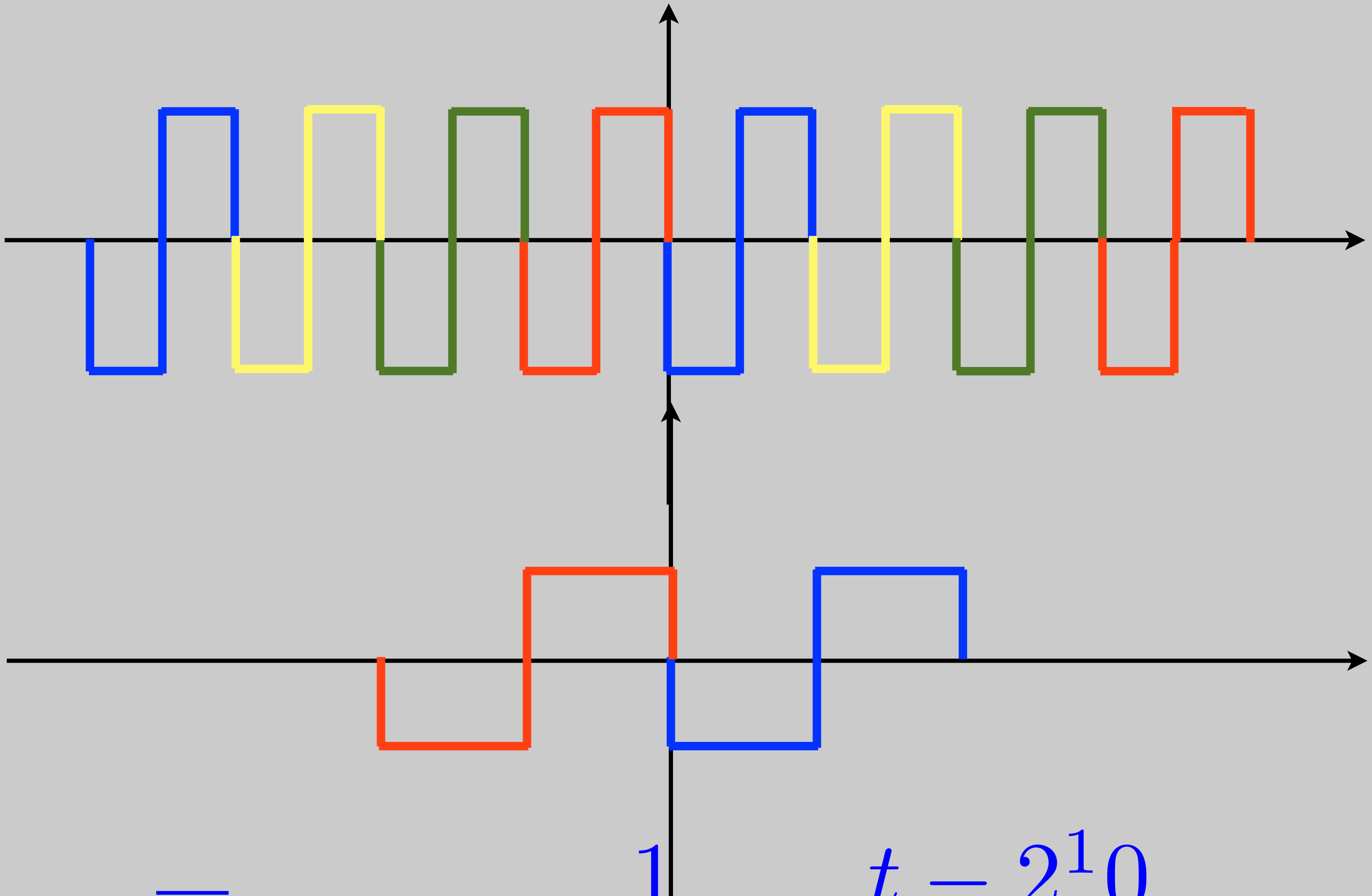
Same scale non-overlapping



Orthogonal between scales

$$\overline{\Psi}_{1,0}(t) = \frac{1}{\sqrt{2^1}}\Psi(\frac{t+2^1 0}{2^1}) = \frac{1}{\sqrt{2}}\Psi(\frac{t}{2})$$

Orthonormal Haar



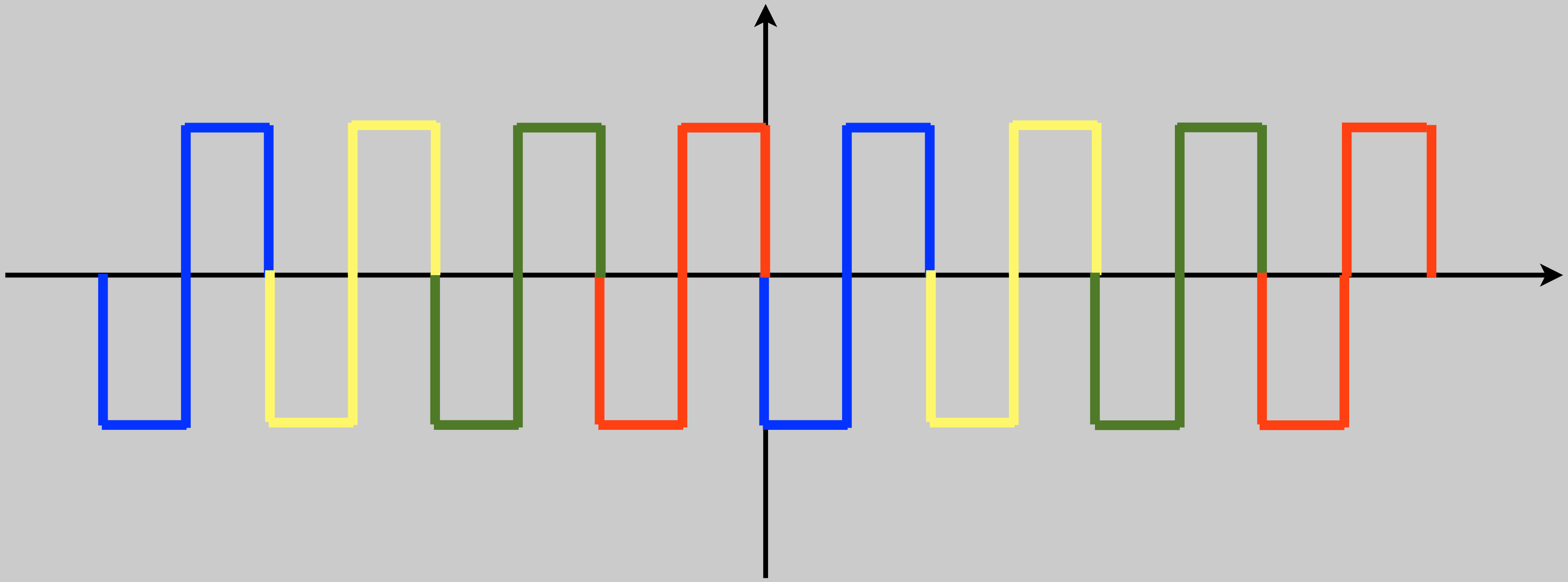
Same scale non-overlapping

Orthogonal between scales

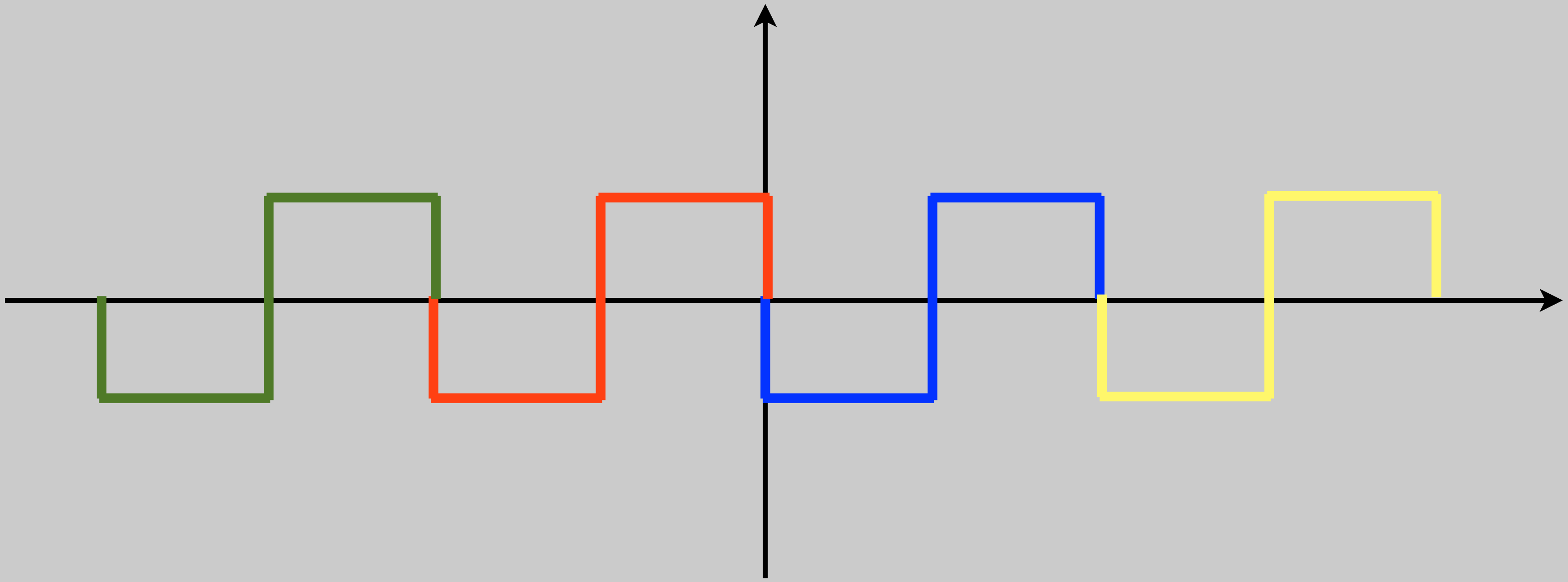
$$\overline{\Psi}_{1,0}(t) = \frac{1}{\sqrt{2^1}}\Psi(\frac{t-2^1 0}{2^1}) = \frac{1}{\sqrt{2}}\Psi(\frac{t}{2})$$

$$\overline{\Psi}_{1,-1}(t) = \frac{1}{\sqrt{2^1}}\Psi(\frac{t+2^1 1}{2^1}) = \frac{1}{\sqrt{2}}\Psi(\frac{t+2}{2})$$

Orthonormal Haar



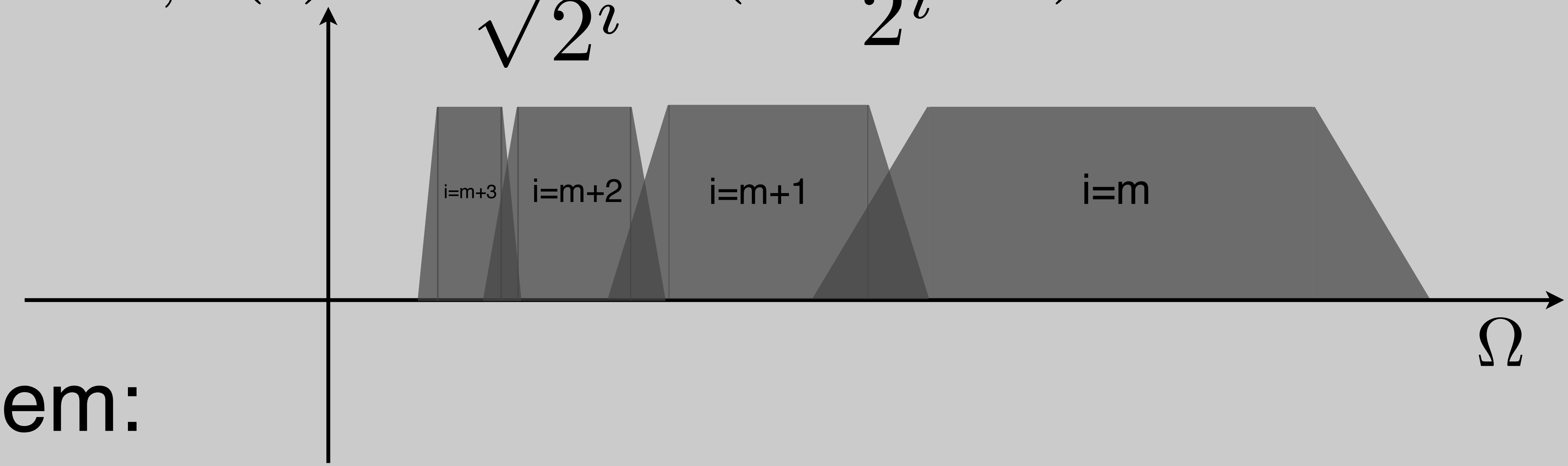
Same scale
non-overlapping



Orthogonal
between scales

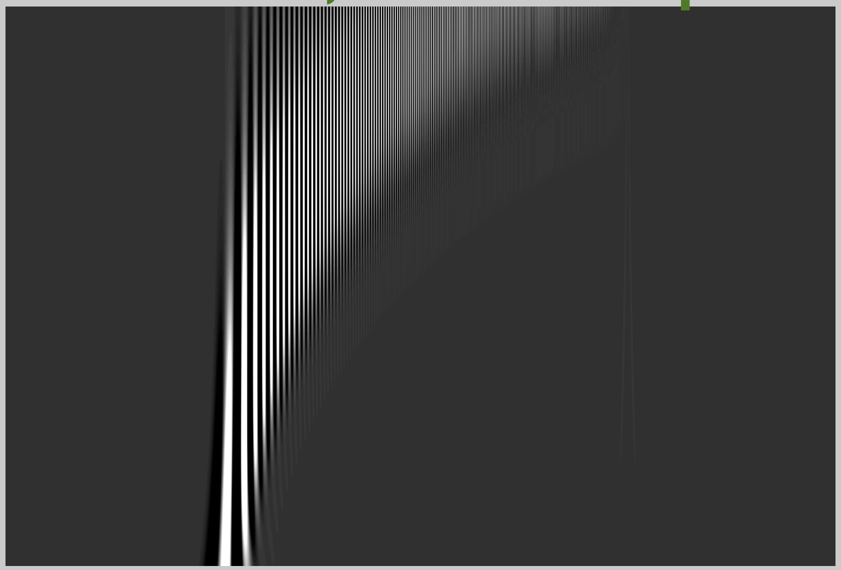
Scaling function

$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$



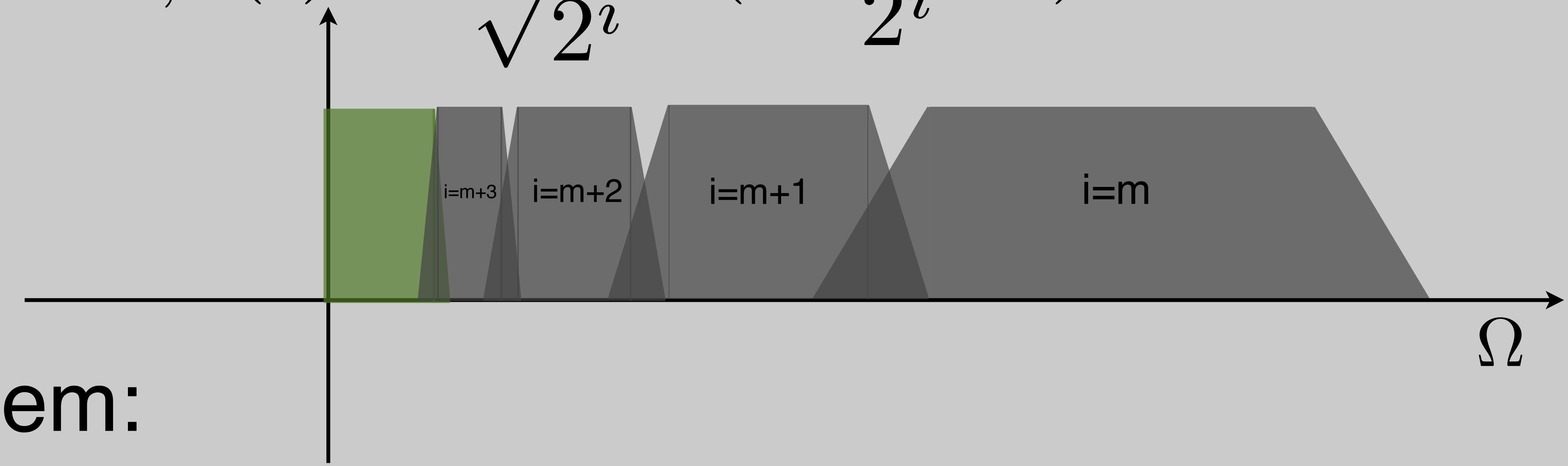
- Problem:
  - –Every stretch only covers half remaining bandwidth
  - –Need Infinite functions

recall, for chirp:



Scaling function

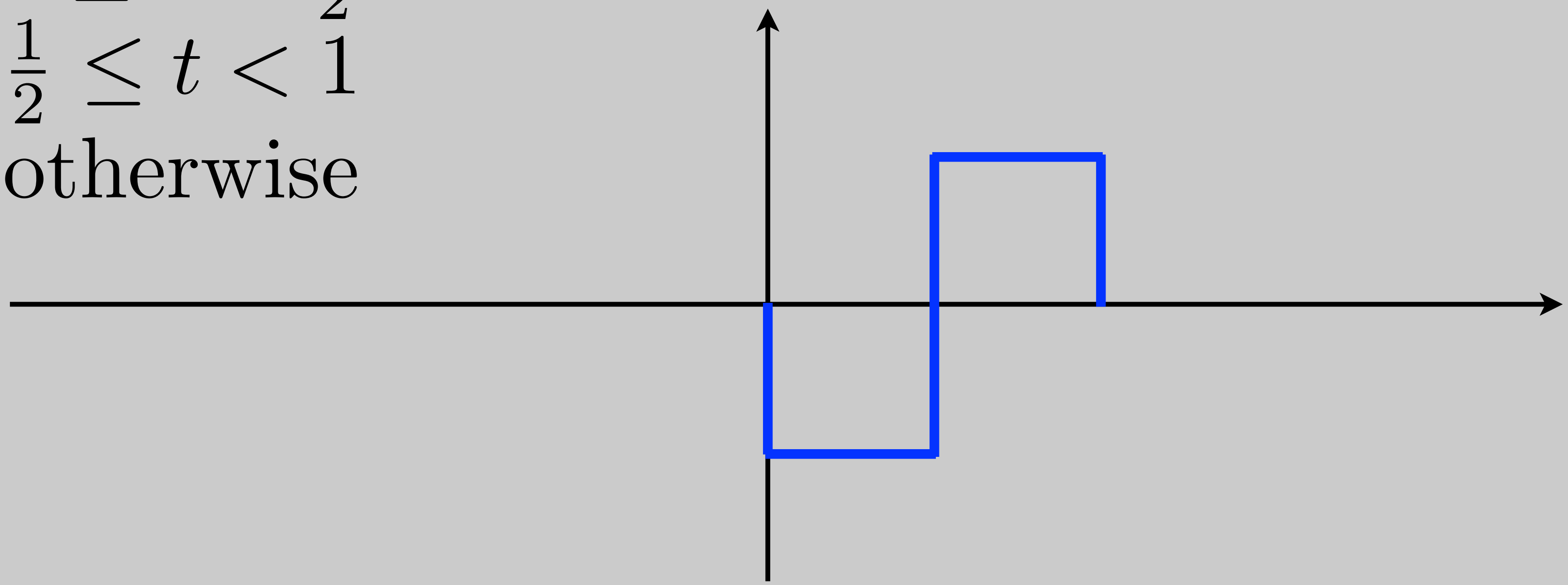
$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi\left(\frac{t-2^i n}{2^i}\right)$$



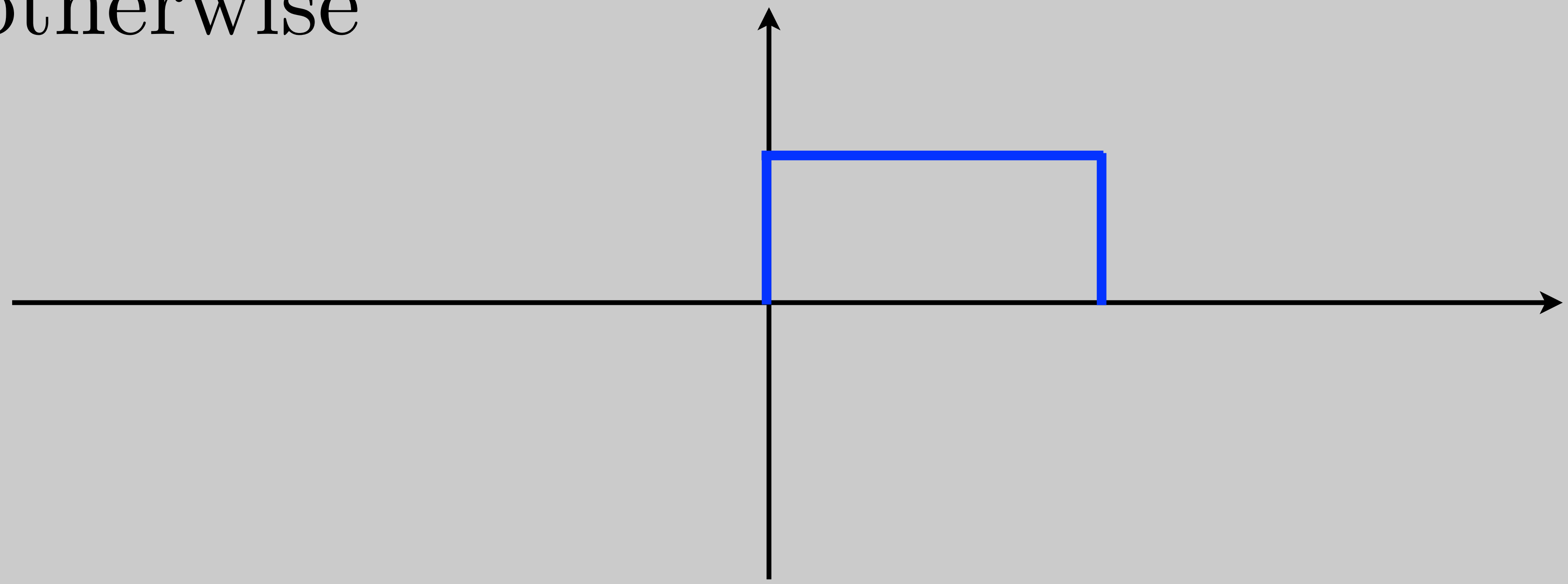
- Problem:
  - –Every stretch only covers half remaining bandwidth
  - –Need Infinite functions
- Solution:
  - –Plug low-pass spectrum with a scaling function $\overline{\Phi}$

Haar Scaling function

$$\Psi(t) = \begin{cases} -1 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\Phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



Back to Discrete

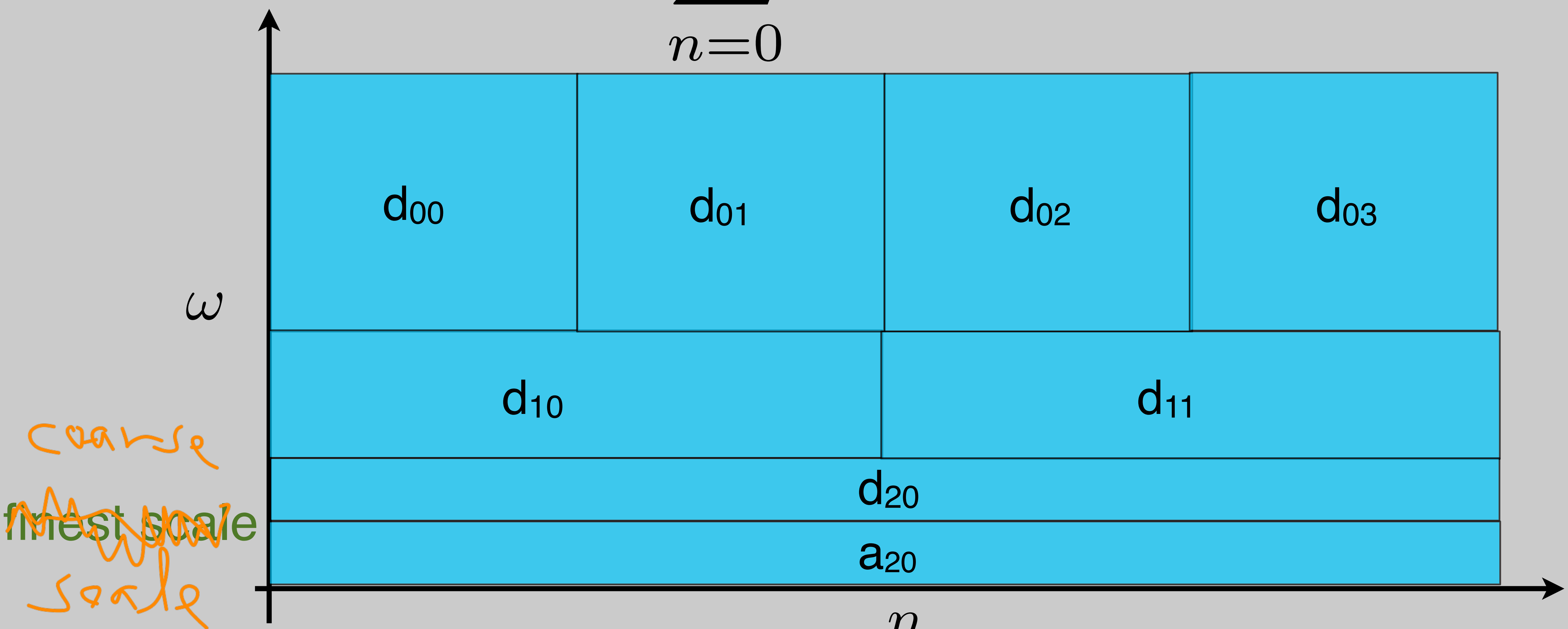
- Early 80's, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
- Late 80's link to DSP by Daubechies and Mallat.

- From CWT to DWT not so trivial!
- Must take care to maintain properties

Discrete Wavelet Transform

$$d_{\mathrm{s,u}} = \sum_{n=0}^{N-1} x[n]\Psi_{\mathrm{s,u}}[n]$$

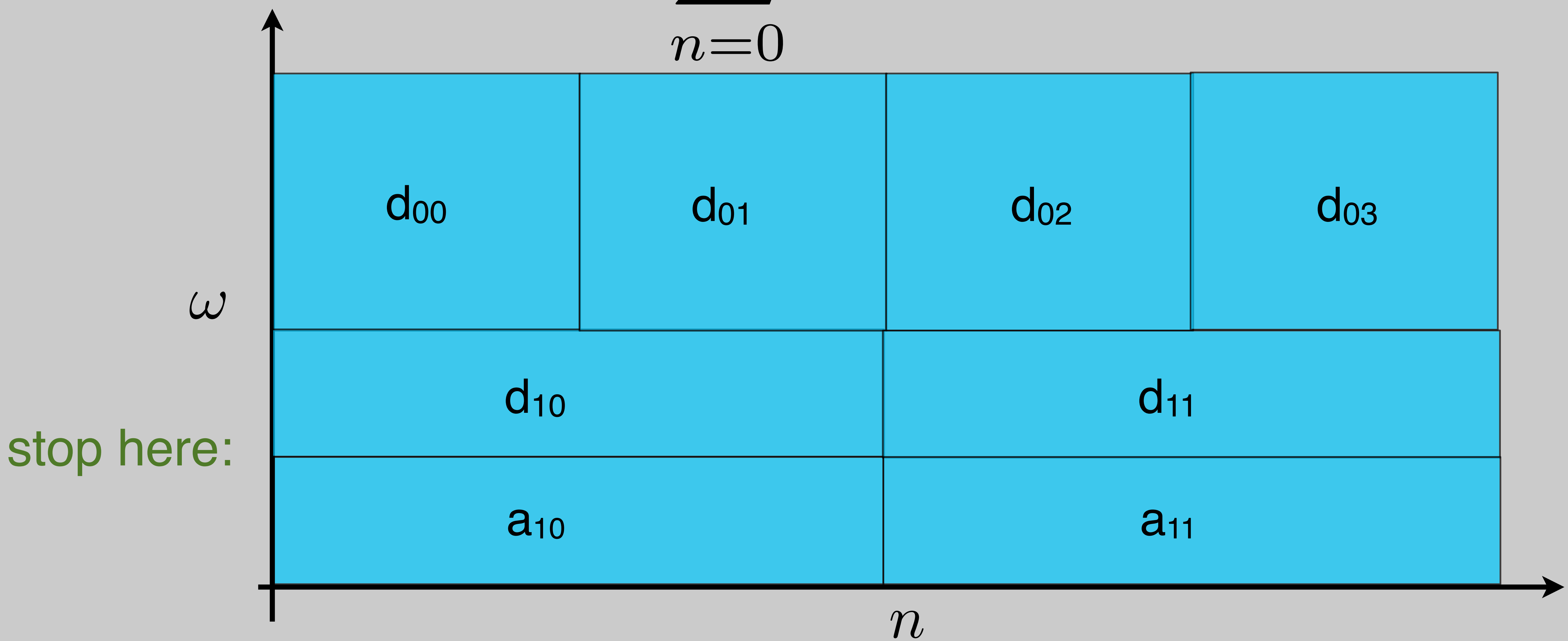
$$a_{\mathrm{s,u}} = \sum_{n=0}^{N-1} x[n]\Phi_{\mathrm{s,u}}[n]$$



Discrete Wavelet Transform

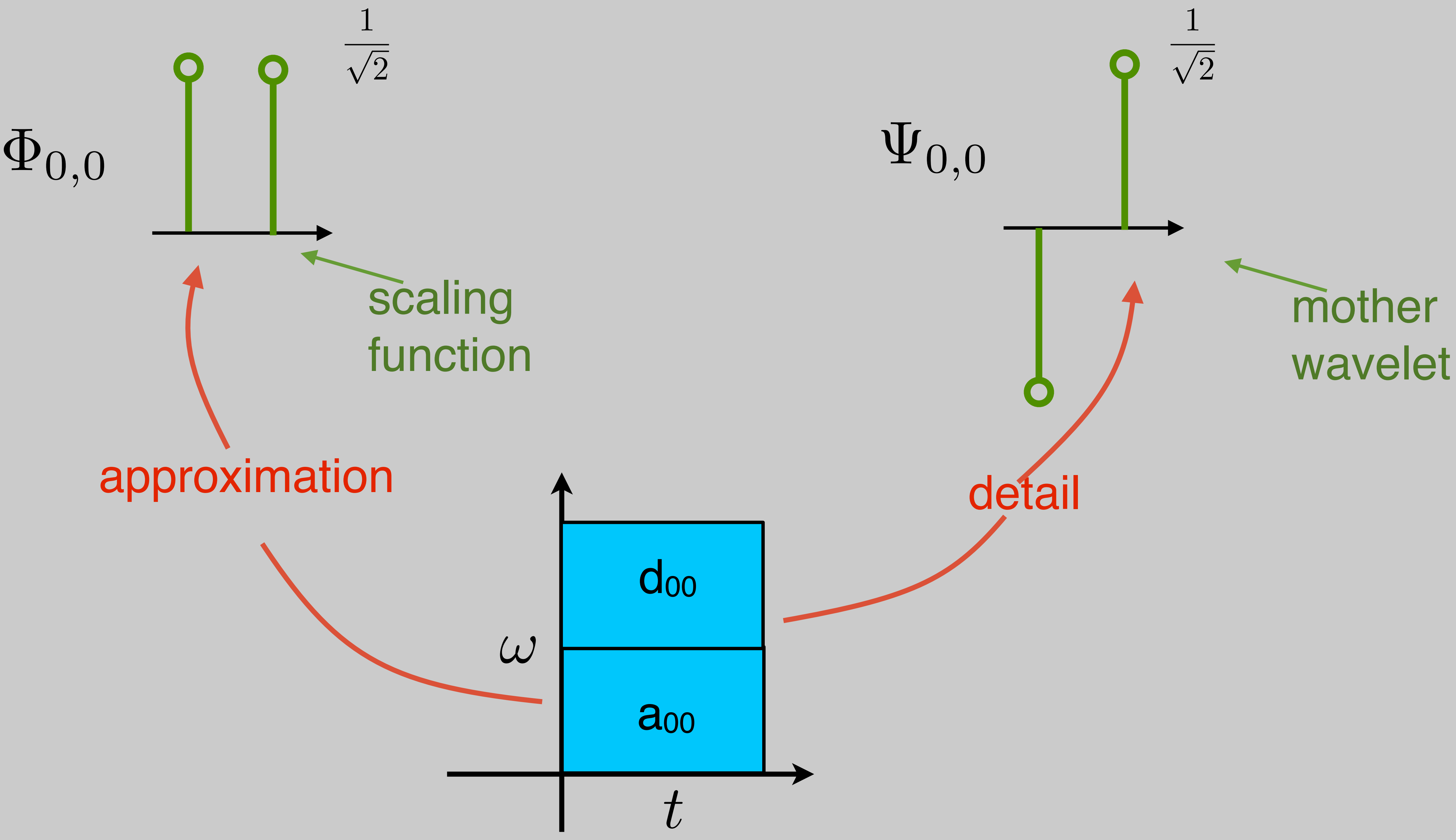
$$d_{\mathrm{s,u}} = \sum_{n=0}^{N-1} x[n]\Psi_{\mathrm{s,u}}[n]$$

$$a_{\mathrm{s,u}} = \sum_{n=0}^{N-1} x[n]\Phi_{\mathrm{s,u}}[n]$$



Example: Discrete Haar Wavelet

Haar for n=2

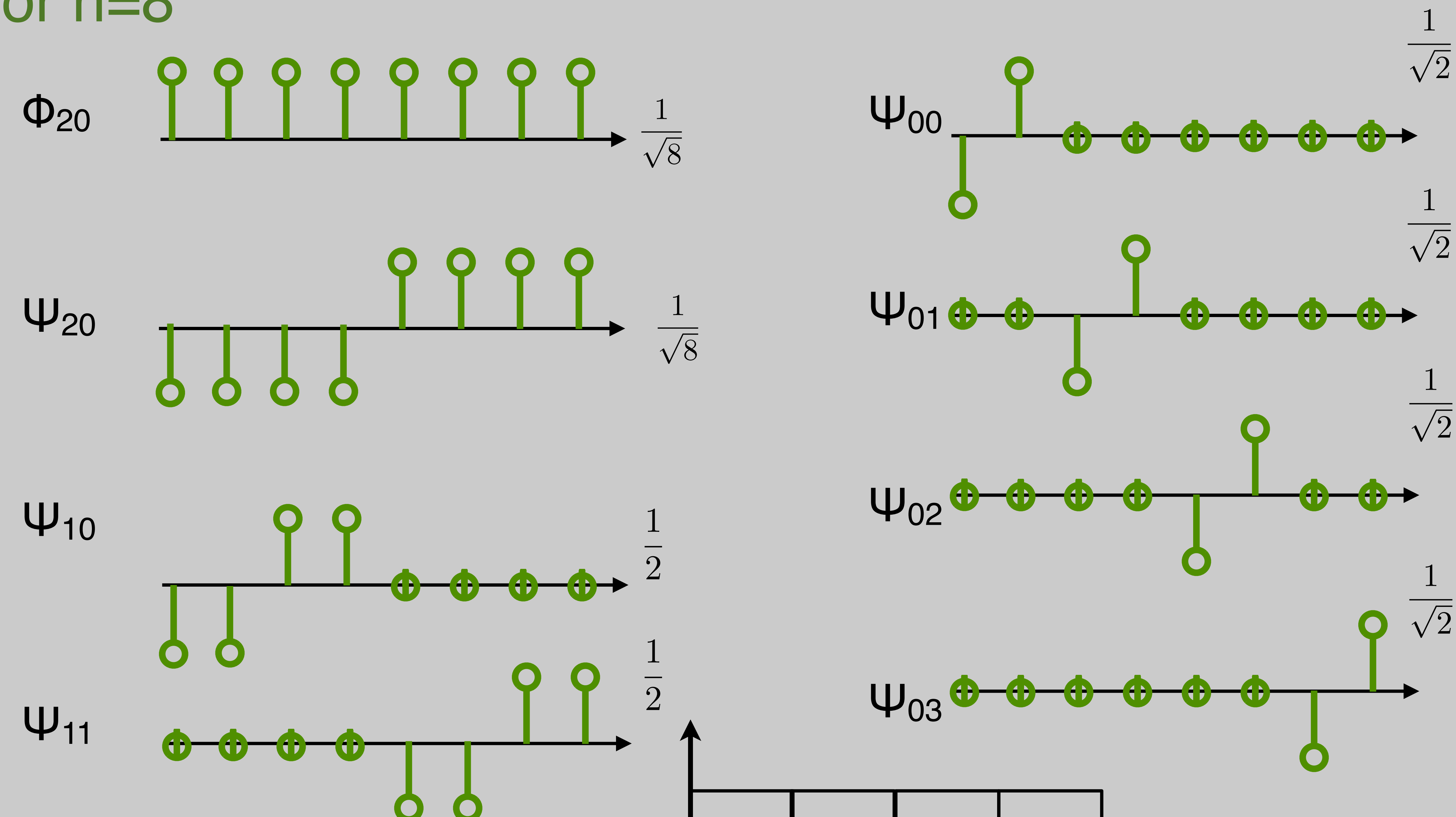


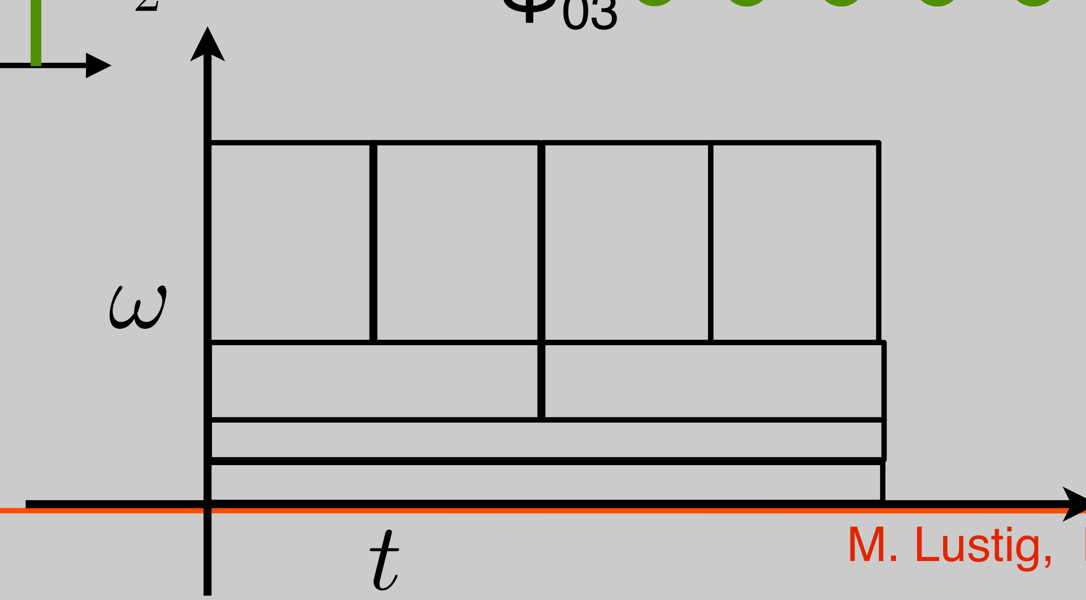
Equivalent to DFT_2!

Discrete Orthogonal Haar Wavelet

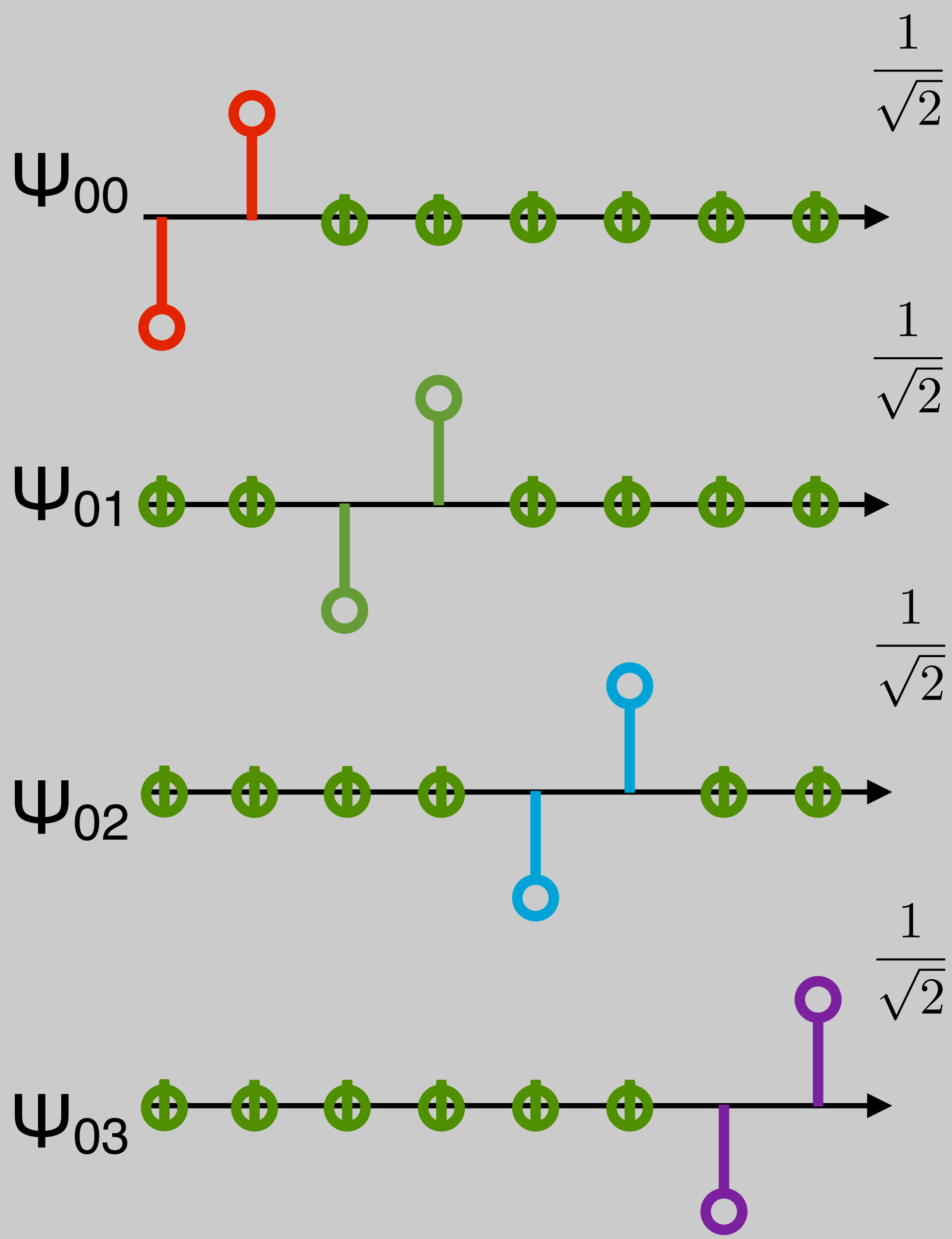
Haar for n=8

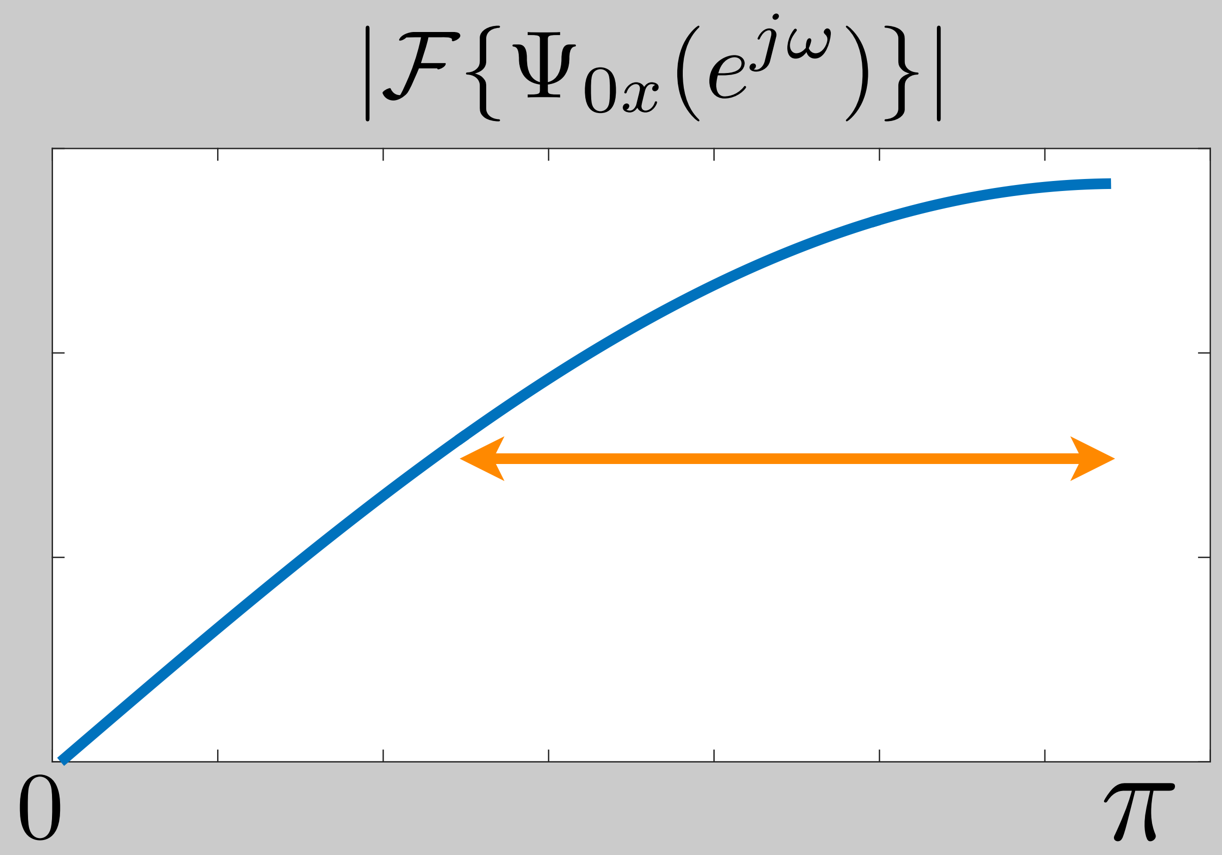
scaling

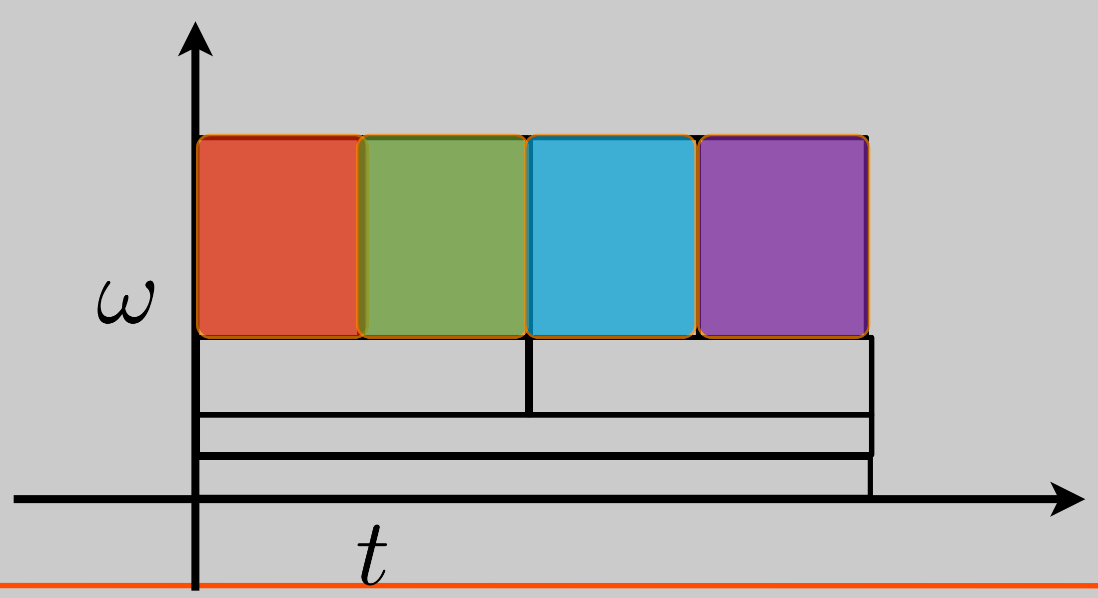




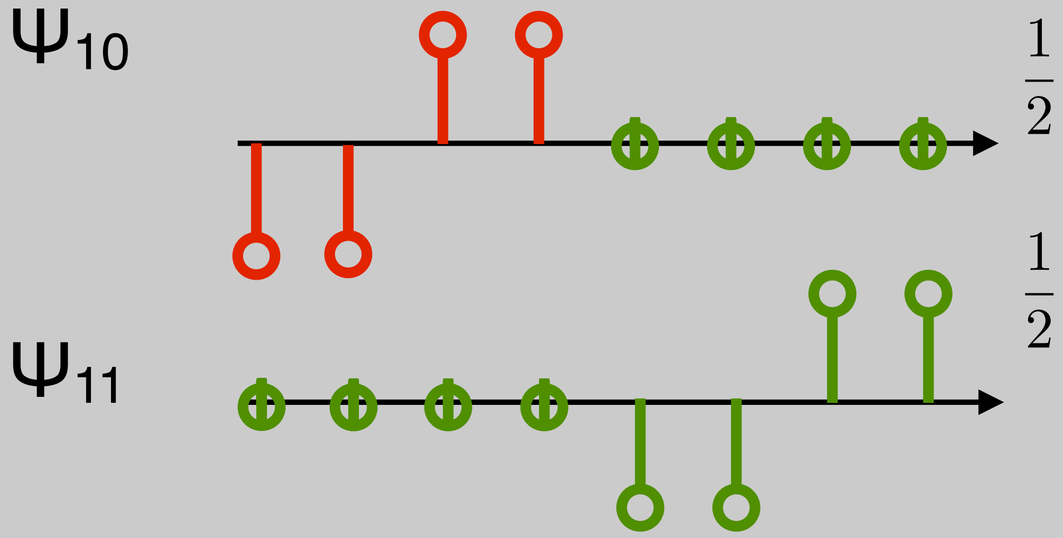
Discrete Orthogonal Haar Wavelet

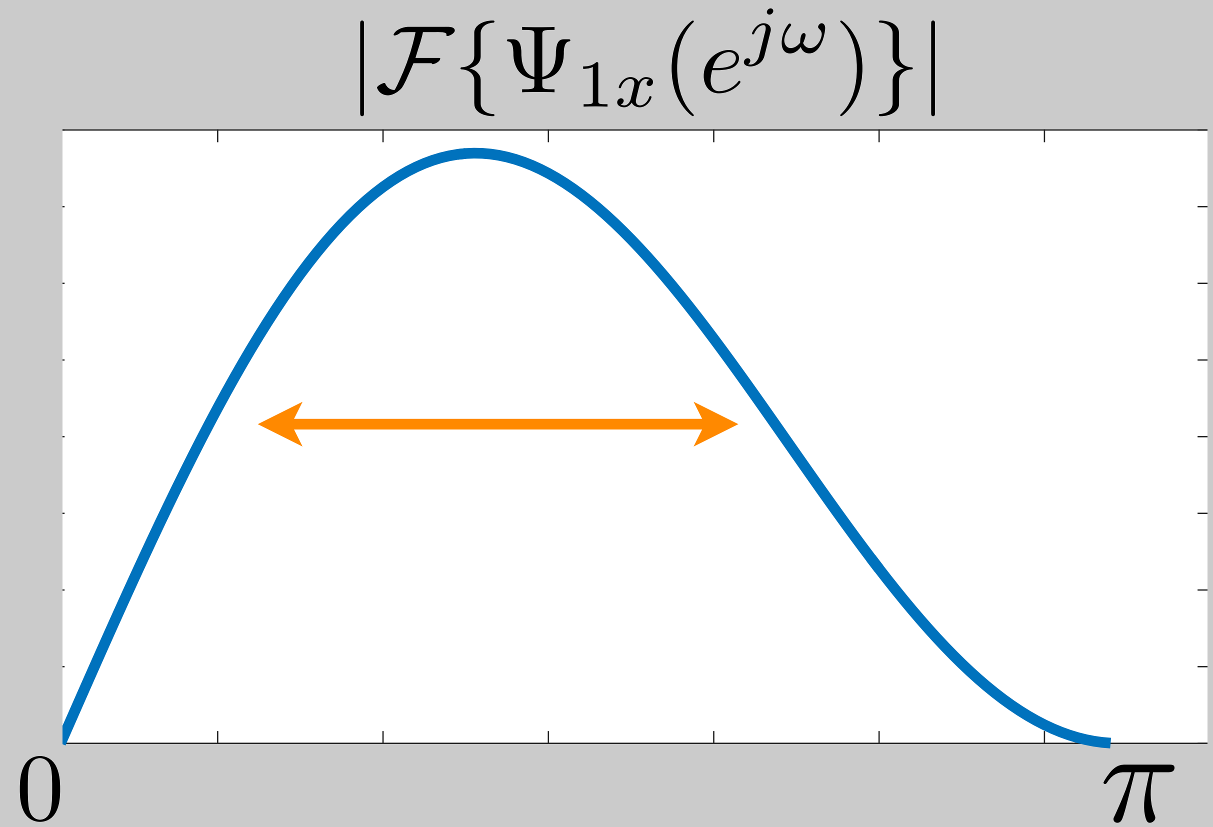


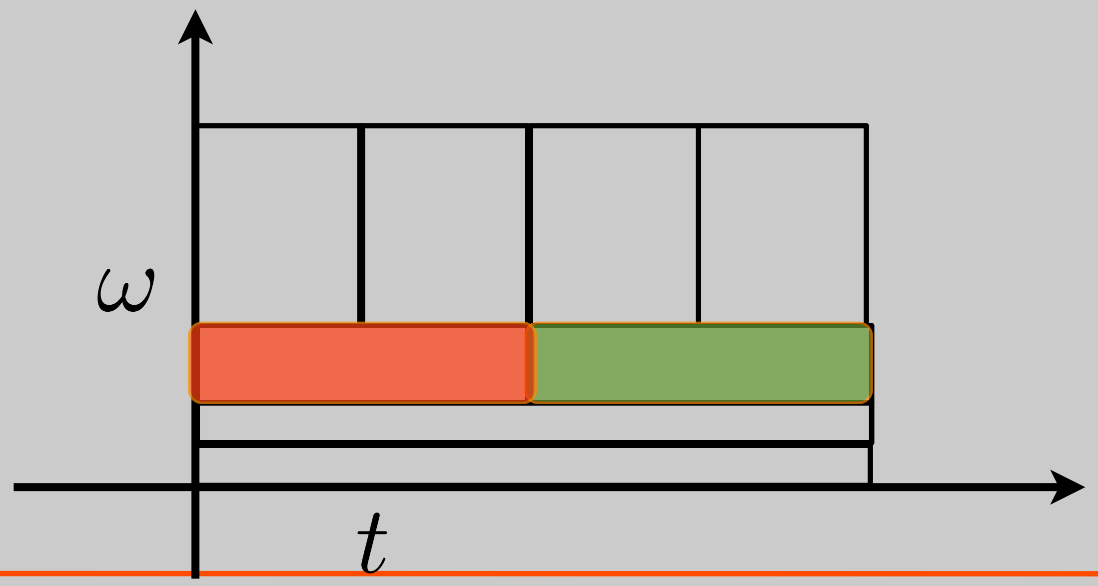




Discrete Orthogonal Haar Wavelet



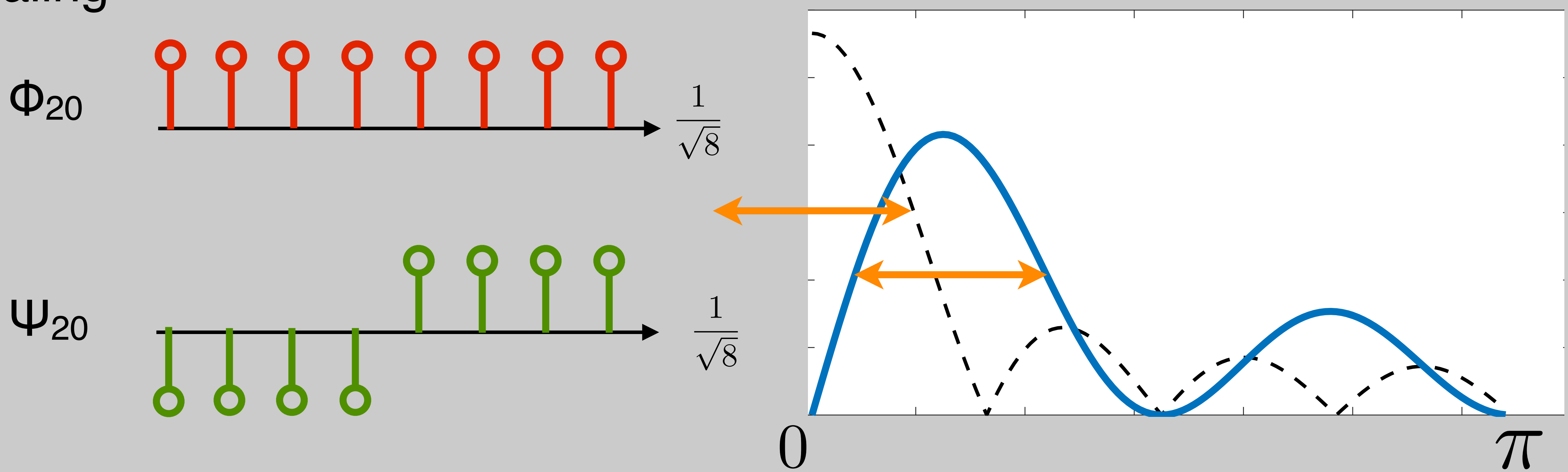


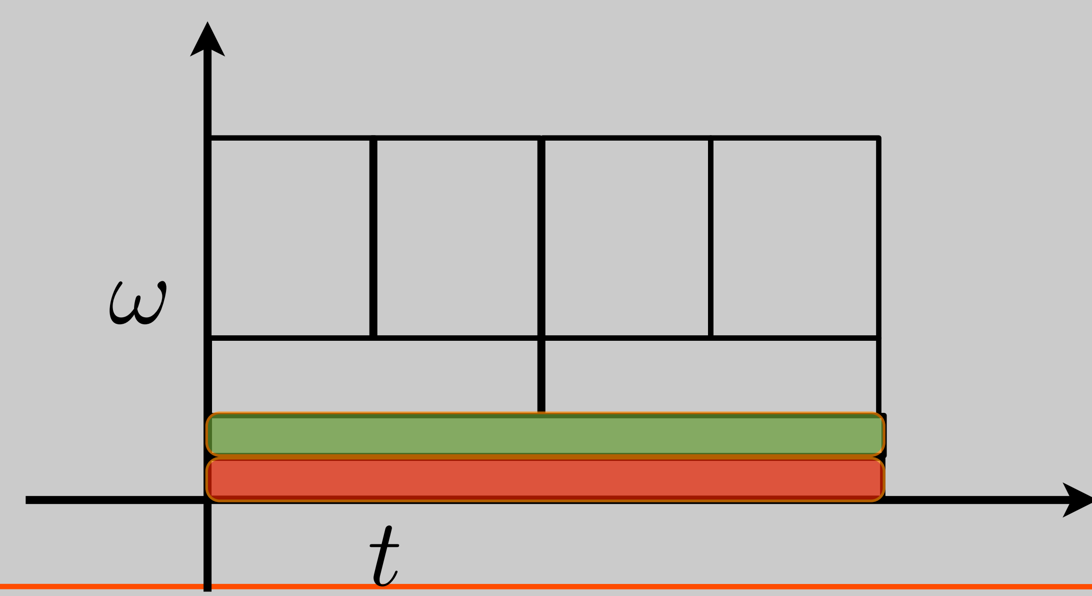


Discrete Orthogonal Haar Wavelet

scaling

$|\mathcal{F}\{\Phi_{2x}(e^{j\omega})\}|$ $|\mathcal{F}\{\Psi_{2x}(e^{j\omega})\}|$





Optional: stop decomposition at Level 1

scaling