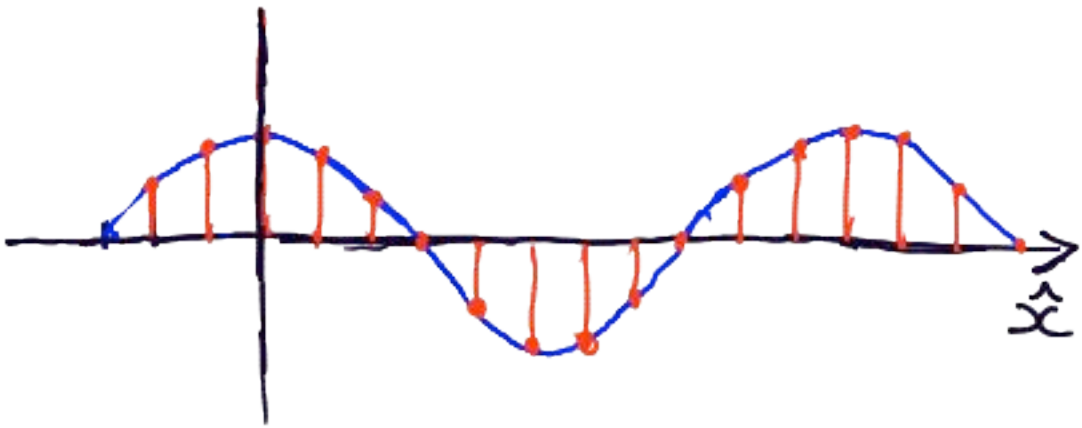
Sampling:

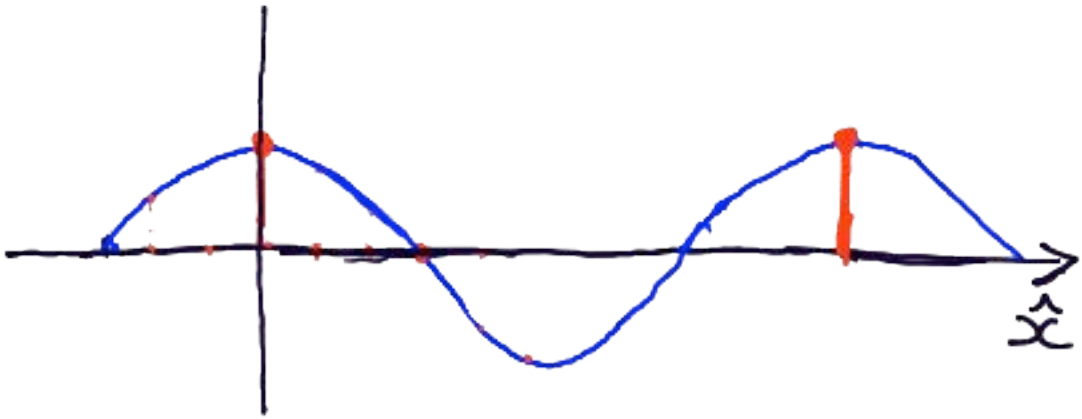
IDEA: How frequently do you need to sample?



LOOKS RIGHT!

Sampling:

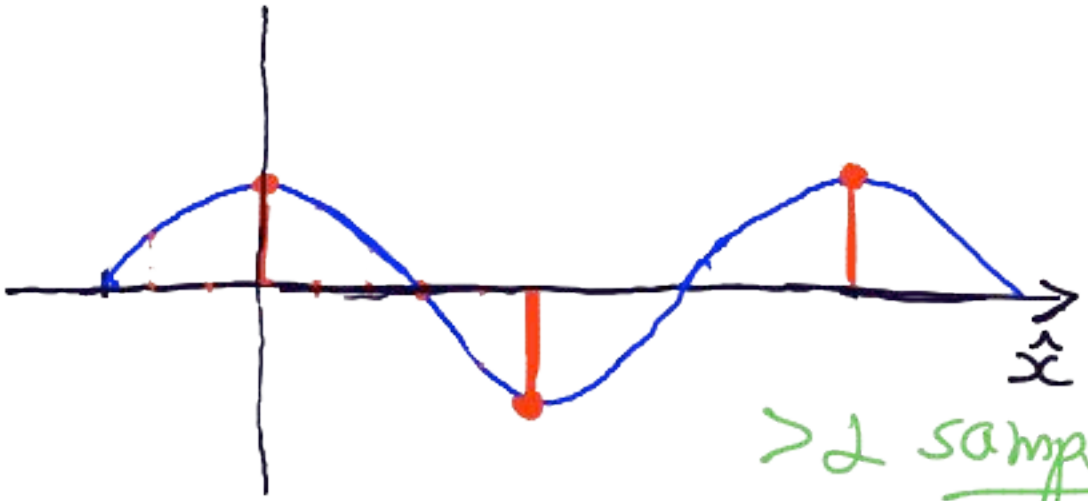
IDEA: How frequently do you need to sample?



Too slow!
(Looks constant)

SAMPLING

#1 is right, but seems over excessive
#2 is too slow
What is the minimum rate?



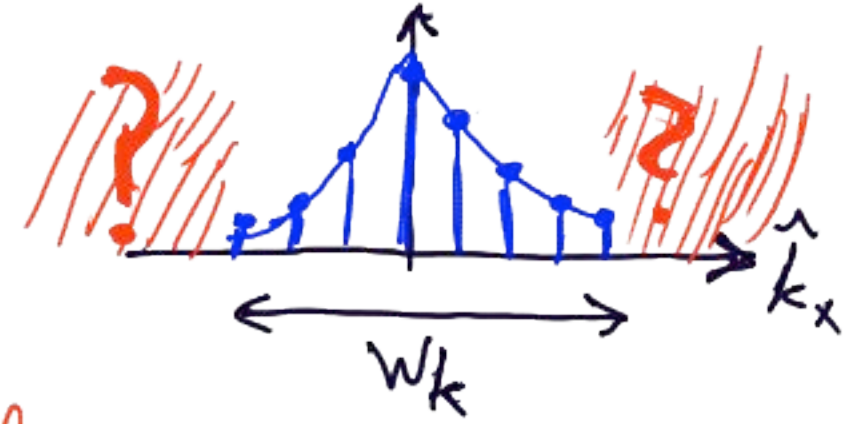
> 2 sample / period

In general rate > 2·BW

show aliasing

sampling in k-space

In MRI $F(k)$ is sampled over a limited range



Problems:

Now → ① DISCRETE SAMPLING

② FINITE SUPPORT

sampling in k-space

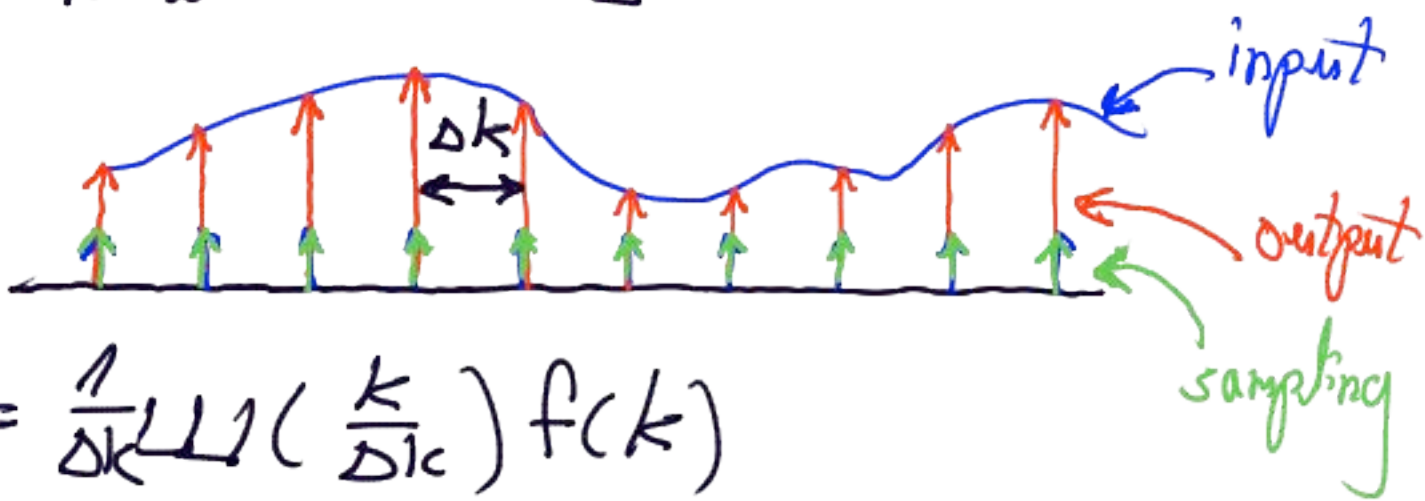
Recall:

$$f(k)\cdot\delta(k-k_0) = f(k_0)\delta(k-k_0)$$

1 sample @ $f(k_0)$

Model sampling using impulse train:

$$\left[\sum_{k=-\infty}^{\infty}\delta(k-n\Delta k)\right] f(k) =$$



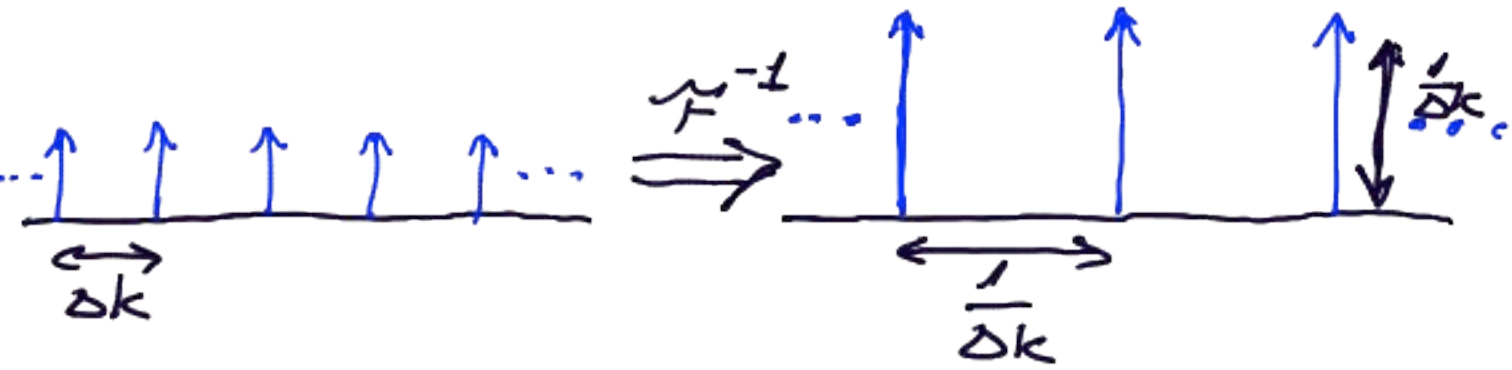
$$= \frac{1}{\Delta k}\amalg\left(\frac{k}{\Delta k}\right) f(k)$$

Recall:

$$F\left(\frac{k}{a}\right) \overset{\mathcal{F}^{-1}}{\Rightarrow} |a| f(ax)$$

So,

$$\frac{1}{\Delta k} \mathrm{III}\left(\frac{k}{\Delta k}\right) \overset{\mathcal{F}^{-1}}{\Longrightarrow} \mathrm{III}(\Delta k x)$$



also recall

$$F(k) \cdot G(k) \overset{\mathcal{F}^{-1}}{\Longrightarrow} f(x) * g(x)$$

So, sampled

if $\hat{F}(k) = F(k) \cdot \frac{1}{\Delta k} \amalg\left(\frac{k}{\Delta k}\right)$

$\Downarrow \mathcal{F}^{-1}$

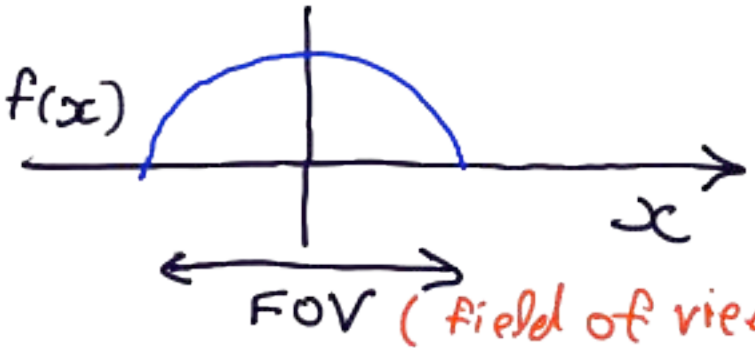
then

$$\hat{f}(x) = \mathcal{F}^{-1}\{\hat{F}(k)\} = f(x) * \amalg(\Delta k x)$$

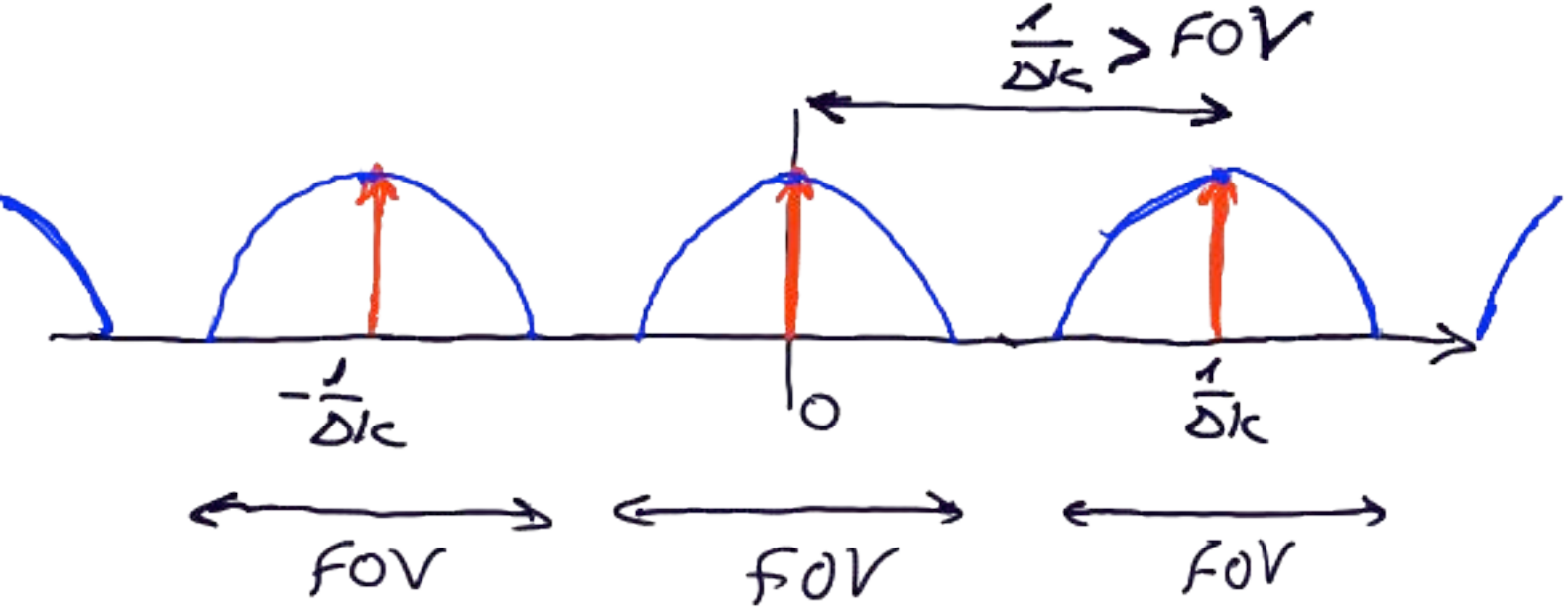
Recall:

$$f(x) * \delta(x - x_0) = f(x - x_0)$$

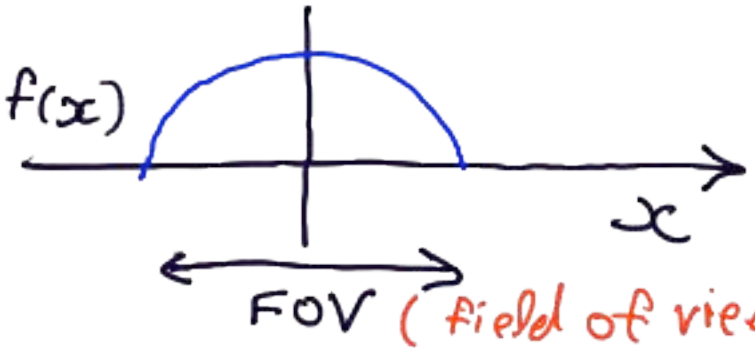
If $f(x)$ is SPACE LIMITED:



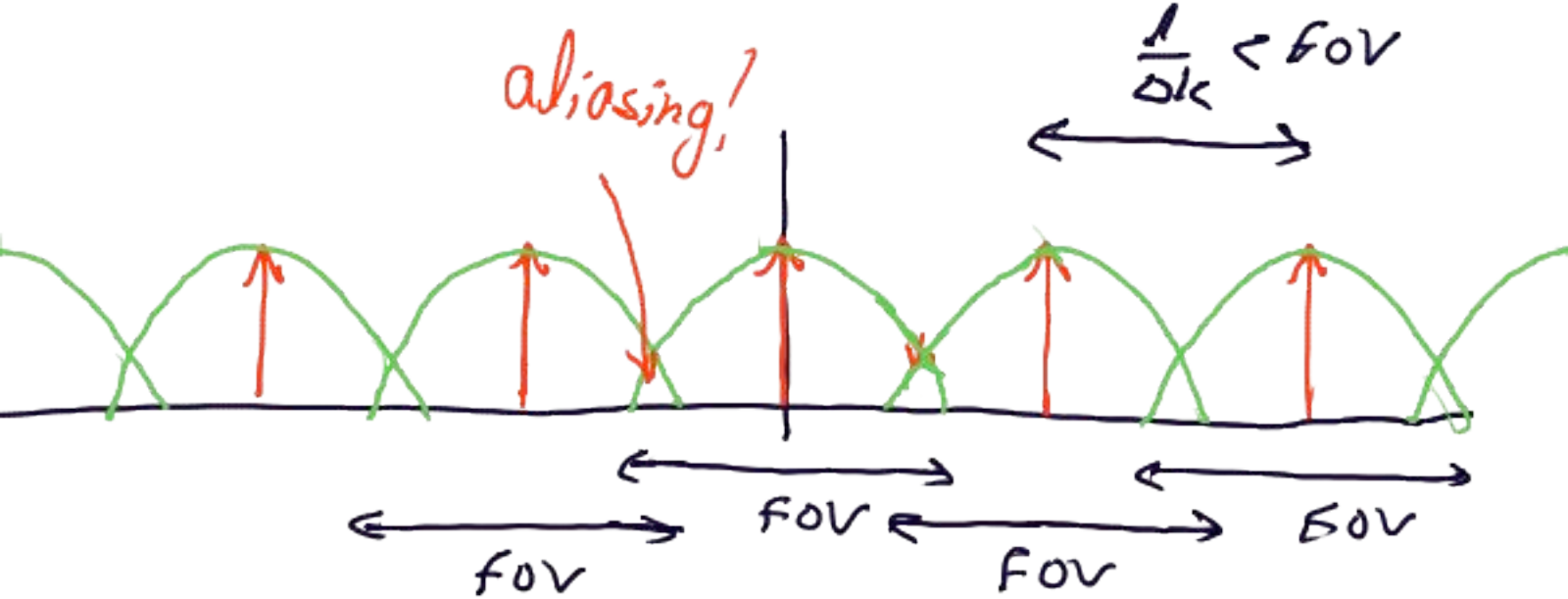
$$\hat{f}(x) = f(x) * \amalg(\Delta k x)$$



If $f(x)$ is SPACE LIMITED:

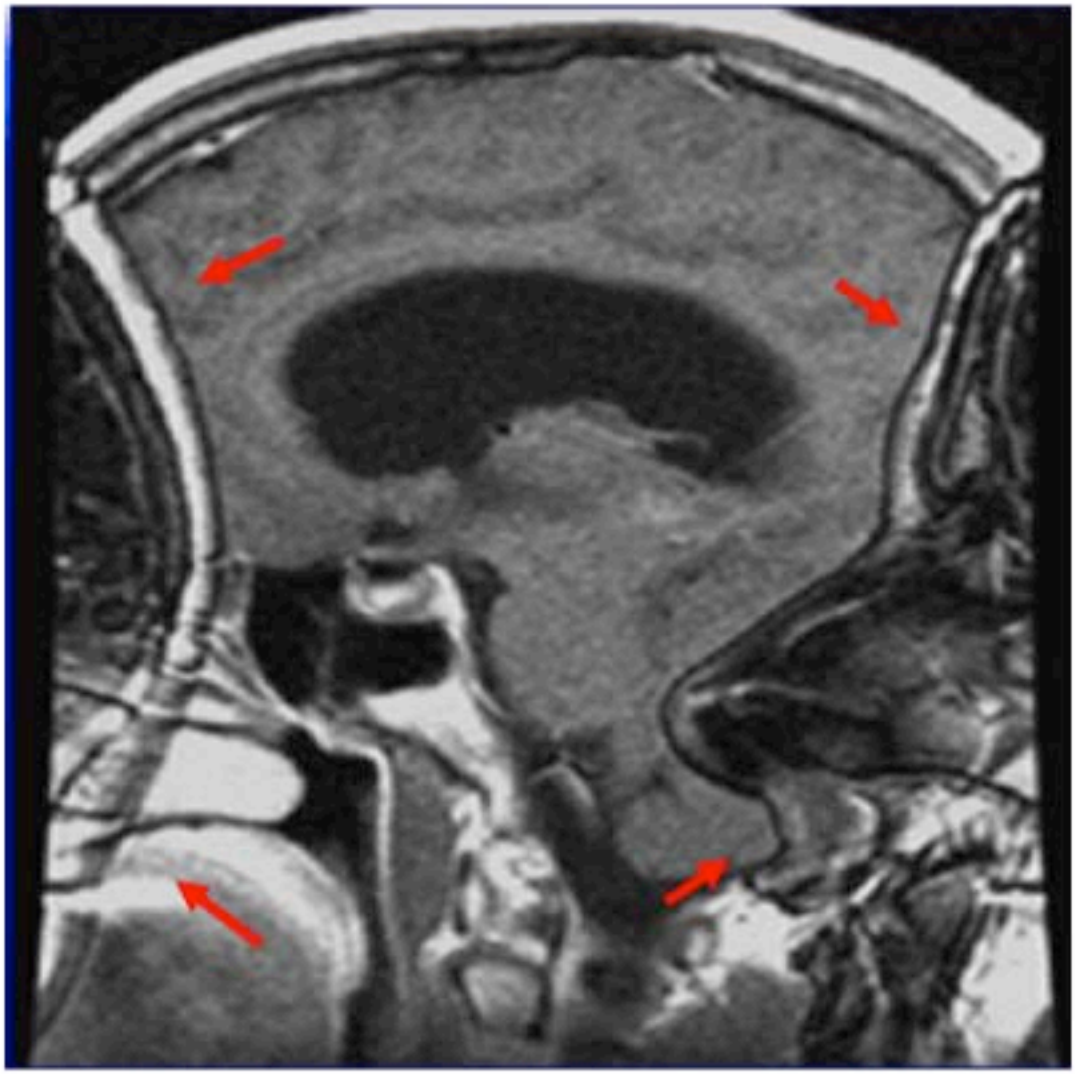


$$\hat{f}(x) = f(x) * \amalg(\Delta k x)$$



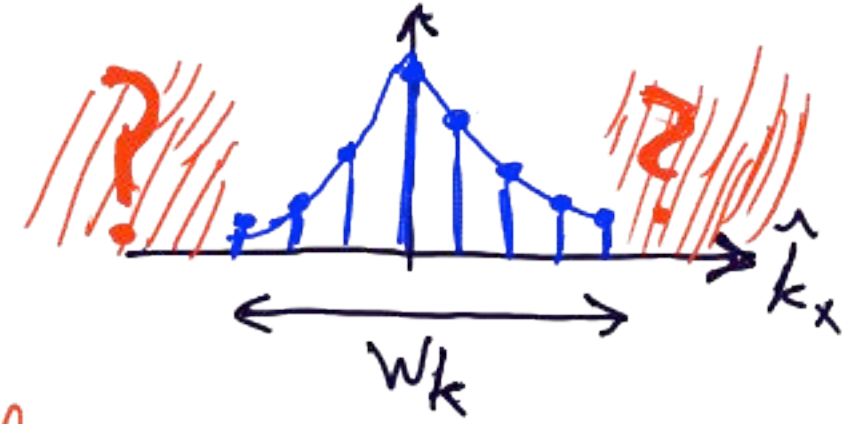
Aliasing in x due to large Δk_x

8a



sampling in k-space

In MRI $F(k)$ is sampled over a limited range



Problems:

① DISCRETE SAMPLING

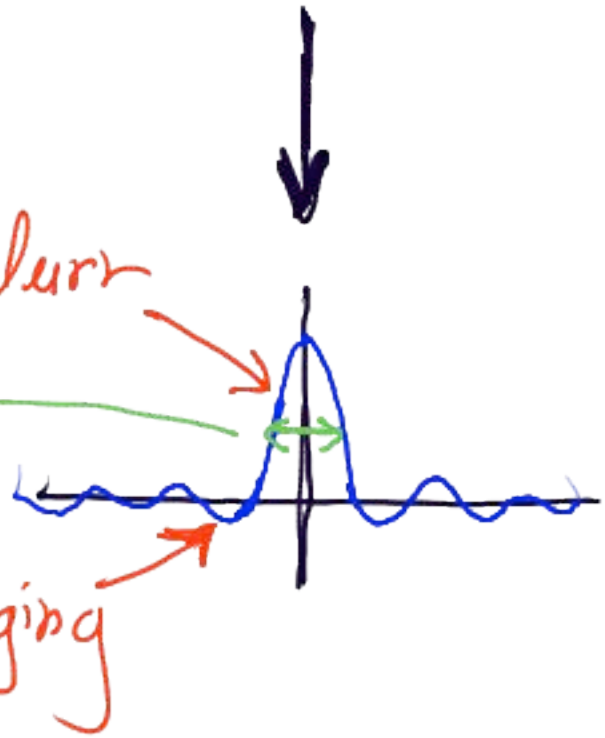
Now → ② FINITE SUPPORT

Finite support sampling

$$F_W(k) = F(k) \sqcap \left(\frac{k}{W_k}\right)$$

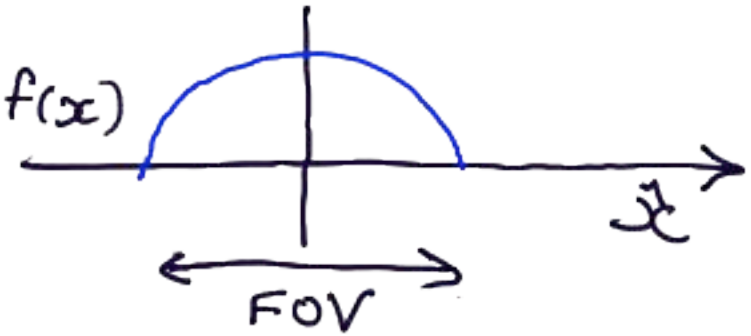
"windowed"

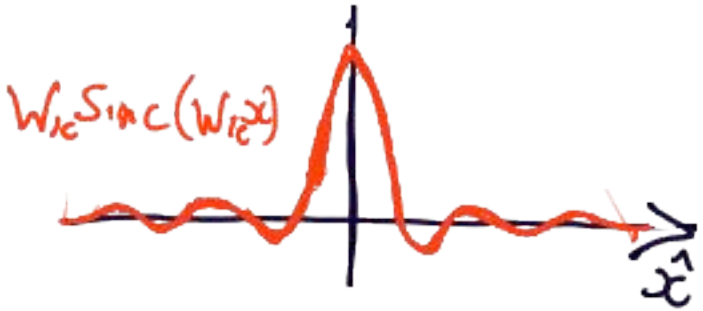
$$\Rightarrow f_W(x) = f(x) * W_k \operatorname{sinc}(W_k x)$$



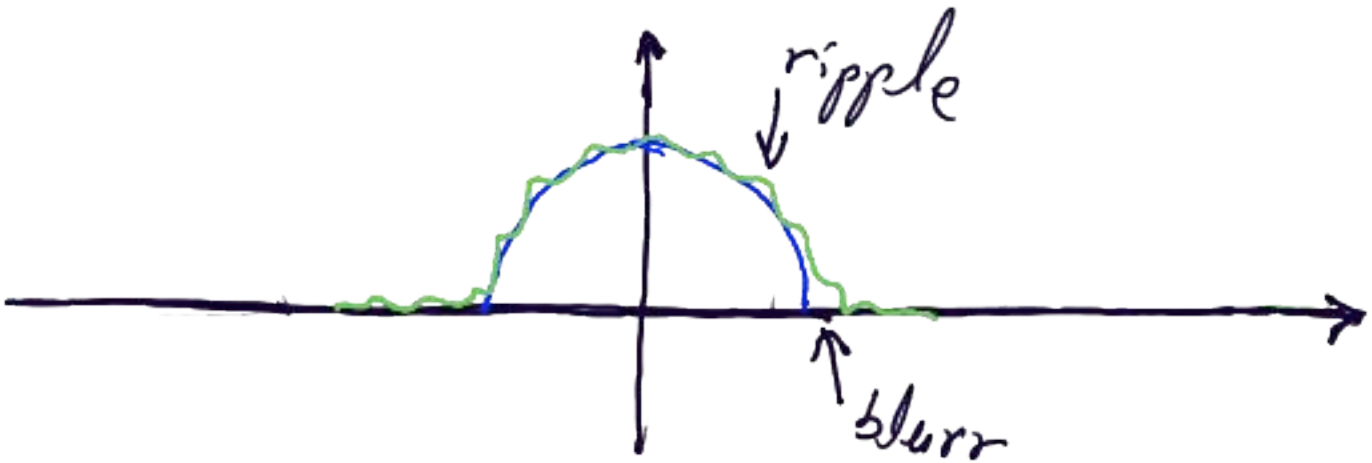
Full width half max (FWHM) $= \frac{1}{W_k}$

If $f(x)$ is





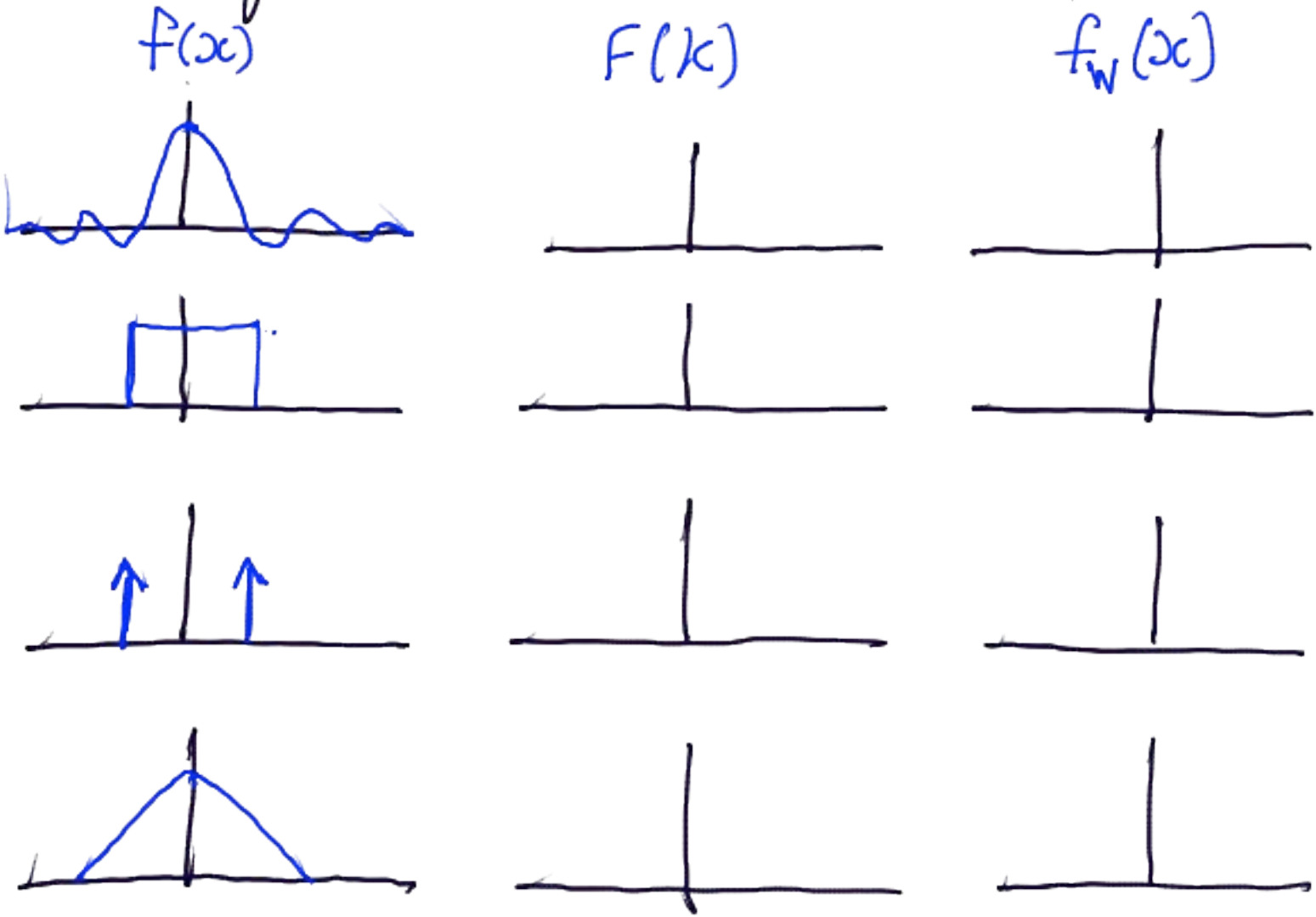
$$f_W(x) = f(x) * W_k \mathrm{sinc}(W_k x)$$



QUIZ

11a

Rank the "strength" of artifact due to k-space truncation



Discrete & finite sampling

Together:

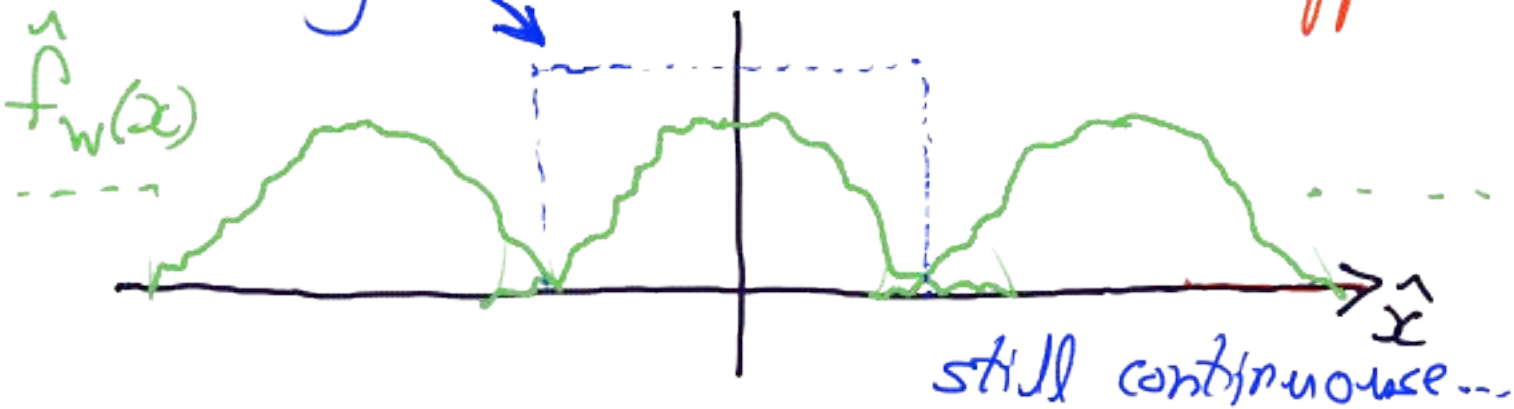
$$\hat{F}_W(k) = F(k) \cdot \frac{1}{\Delta k} \amalg\left(\frac{k}{\Delta k}\right) \Pi\left(\frac{k}{W_k}\right)$$

$$\xrightarrow{\mathcal{F}^{-1}} \quad \hat{f}_W(x) = f(x) * \amalg(\Delta k x) * W_k \operatorname{sinc}(W_k x)$$

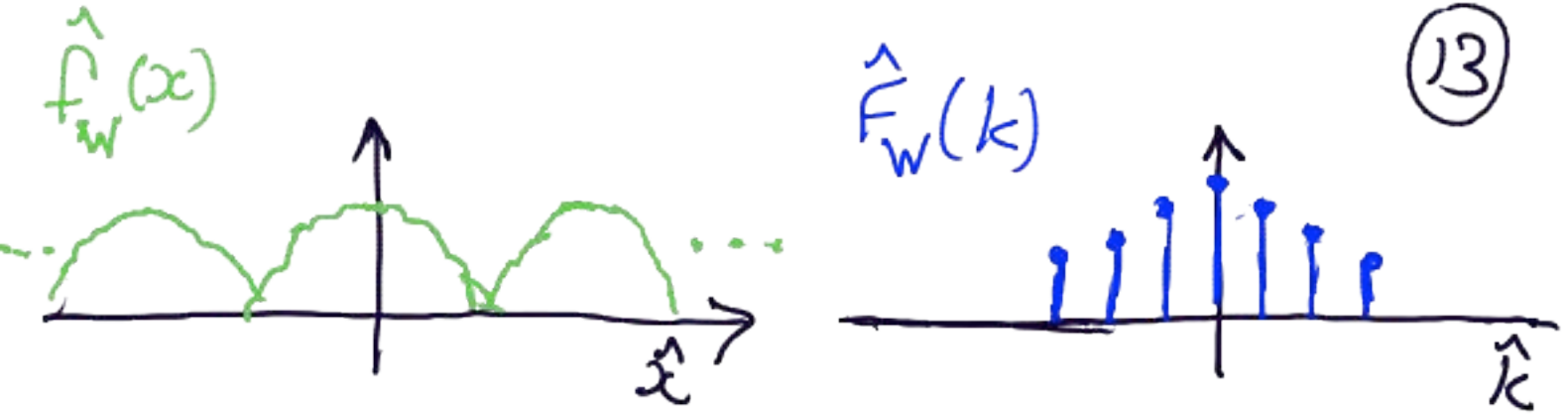
replication

blurr + ripple

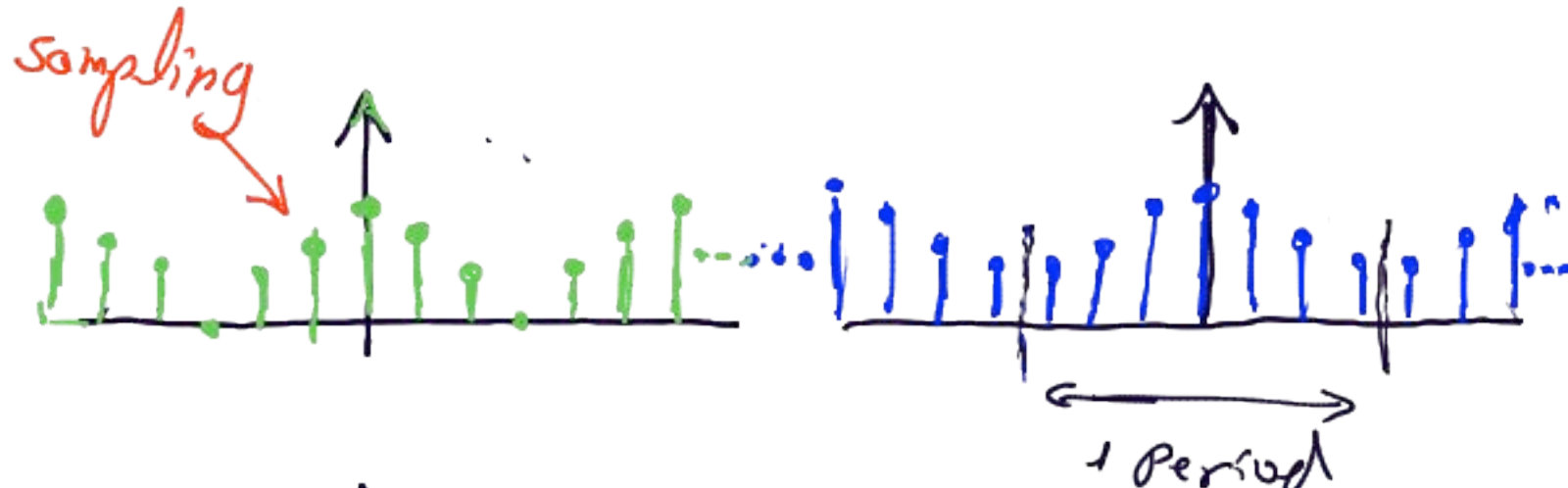
Display



still continuous...



...But images are discrete pixels...
so $\hat{f}_w(x)$ is sampled



$$\hat{f}_w(x) \Longleftrightarrow \hat{F}_w(k)$$

$$f_w[n] \overset{DFT}{\Longleftrightarrow} F_w[m]$$

integer

DFT (centered)

$$F[m] = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} f[n]\, e^{-i 2\pi \frac{mn}{N}}$$

$$f[n] = \frac{1}{N} \sum_{m=-\frac{N}{2}}^{\frac{N}{2}-1} F[m]\, e^{+2\pi \frac{mn}{N}}$$

FAST computation with FFT

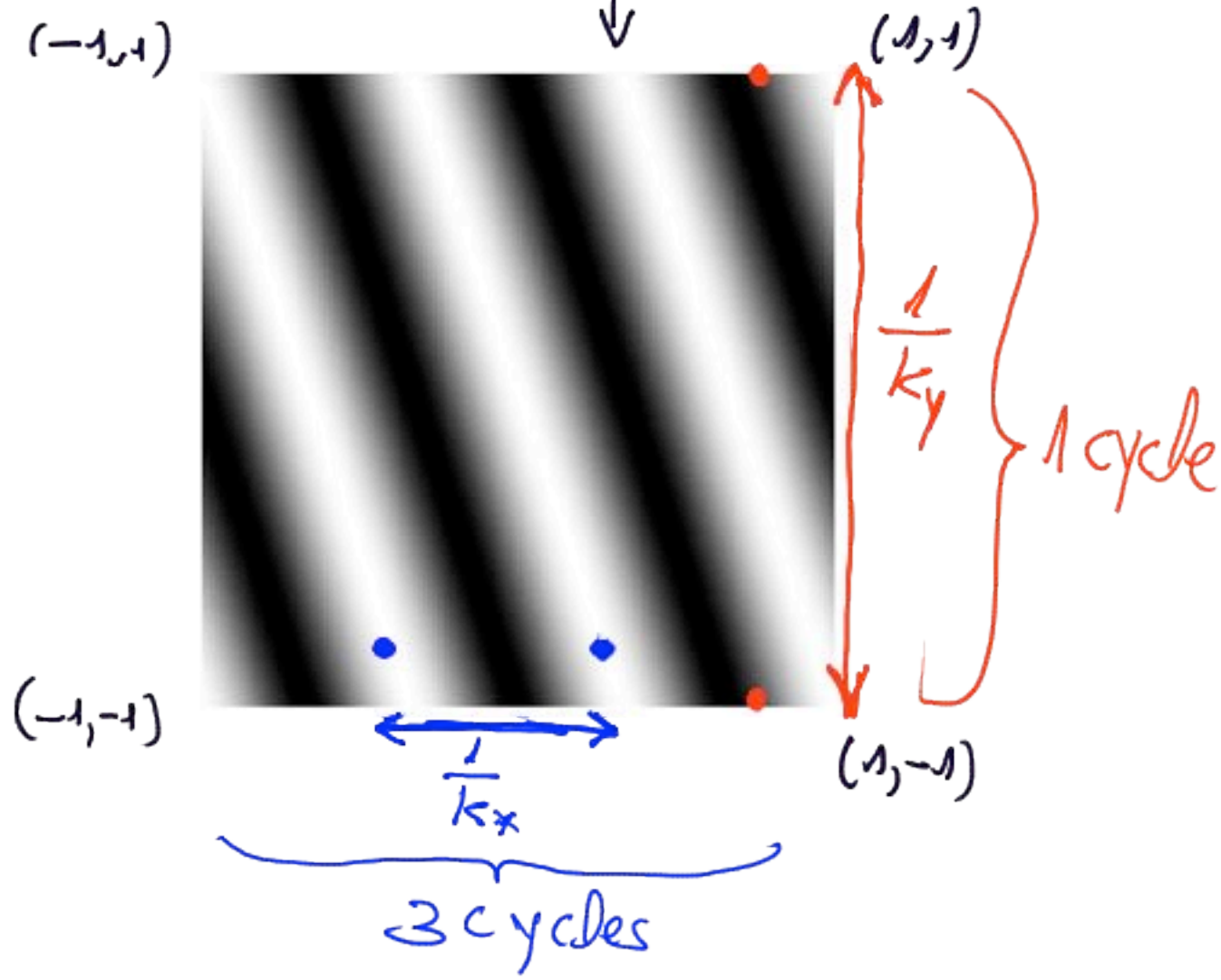
MATLAB:

DFT = fftshift(fft(fftshift))

IDFT = fftshift(ifft(ifftshift))

SPATIAL HARMONICS & 2D FT

This is a spatial harmonic



$$f(x,y) = \cos(2\pi(3x+y))$$

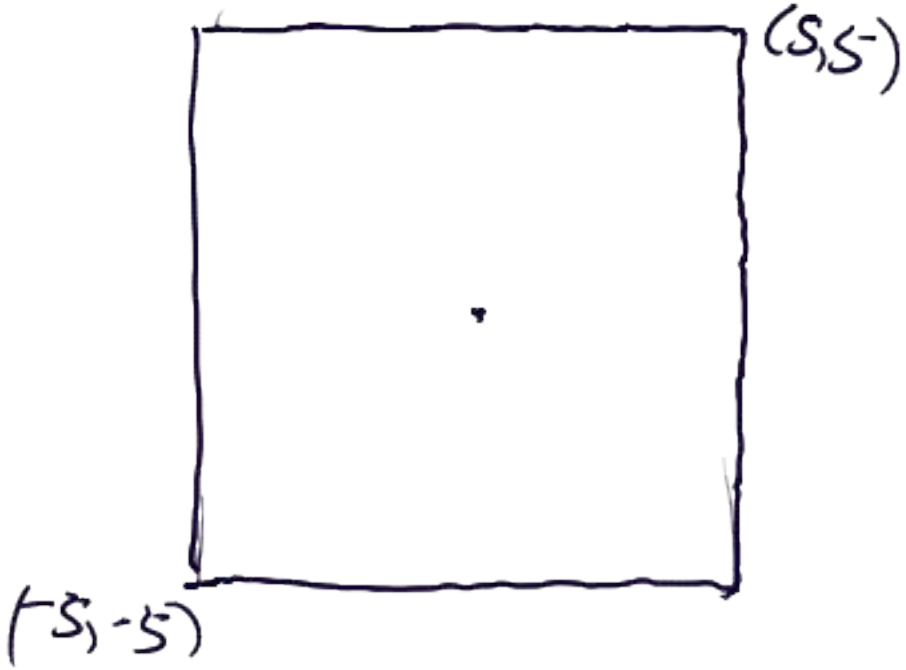
2D FOURIER TRANSFORMS

$$F(k_x,k_y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{-i2\pi(k_x x+k_y y)}\,dx\,dy$$

Q: what is the 2D-FT of

$$f(x,y)=\cos(2\pi(3x+y))$$

A:



seperable functions

$$f(x,y) = f(x)f(y)$$

Q. Which is the 2D-FT?

A. (1) $F(k_x) * F(k_y)$

(2) $F(k_x) \cdot F(k_y)$

sampling & truncation

$$\hat{F_w}(k_x,k_y) = F(k_x,k_y) \cdot \frac{1}{\Delta k_x \Delta k_y} \amalg\left(\frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y}\right) \cdot \Pi\left(\frac{k_x}{W_x}, \frac{k_y}{W_y}\right)$$

aliasing

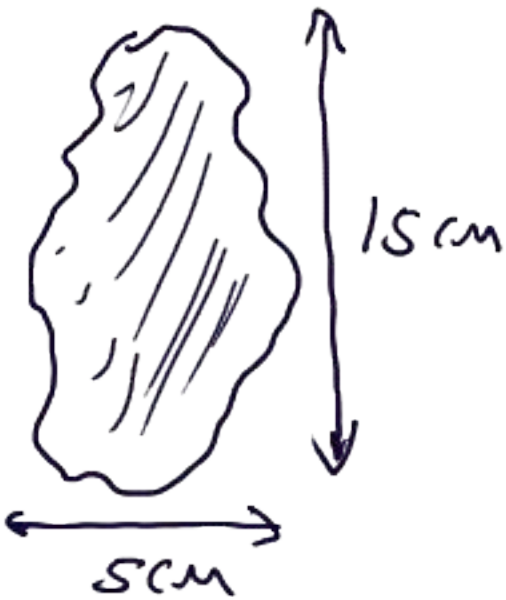
$$\hat{f}_w(x,y) = f(x,y) ** \amalg(\Delta k_x x, \Delta k_y y) ** W_x W_y \operatorname{sinc}(W_x x, W_y y)$$

Blurr

more degrees of freedom for sampling!

2D SAMPLING EXAMPLE

GIVEN $f(x,y)$:



what is Δk_x, Δk_y?

. (to avoid aliasing)

2D SAMPLING EXAMPLE

GIVEN $f(x,y)$:

what is Δk_x, Δk_y?

. (to avoid aliasing)

CIRCULARLY SYMMETRIC

$$f(x,y) = f(r)$$

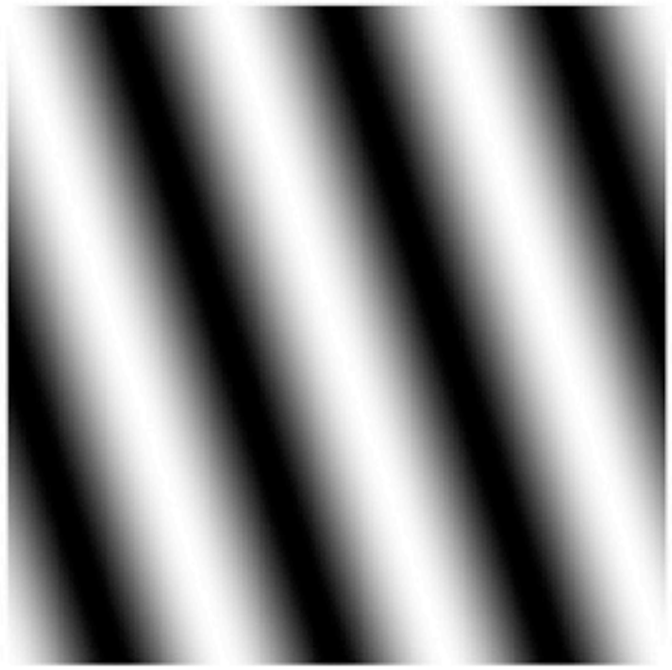
$$\mathcal{F}_{2D}\{f(r)\} = 2\pi \int_0^\infty r f(r) J_0(2\pi r \rho_k)\, dr$$

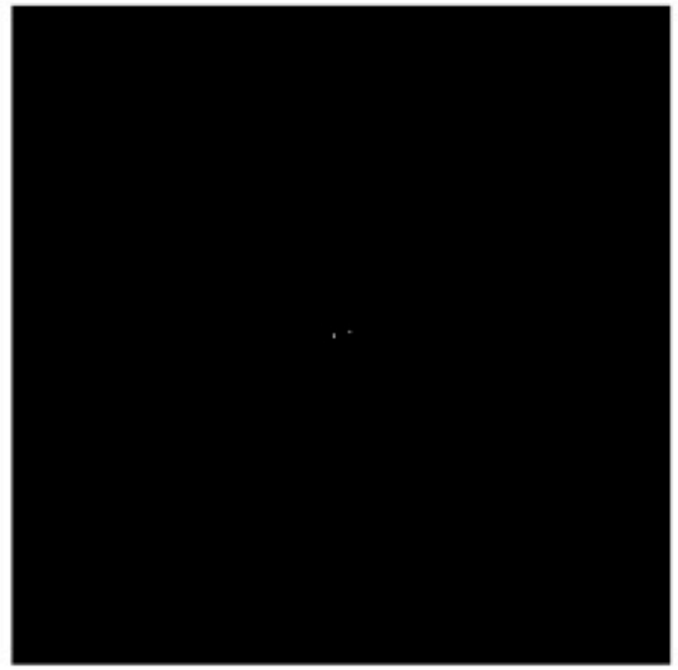
Hankel xform of zero'th order

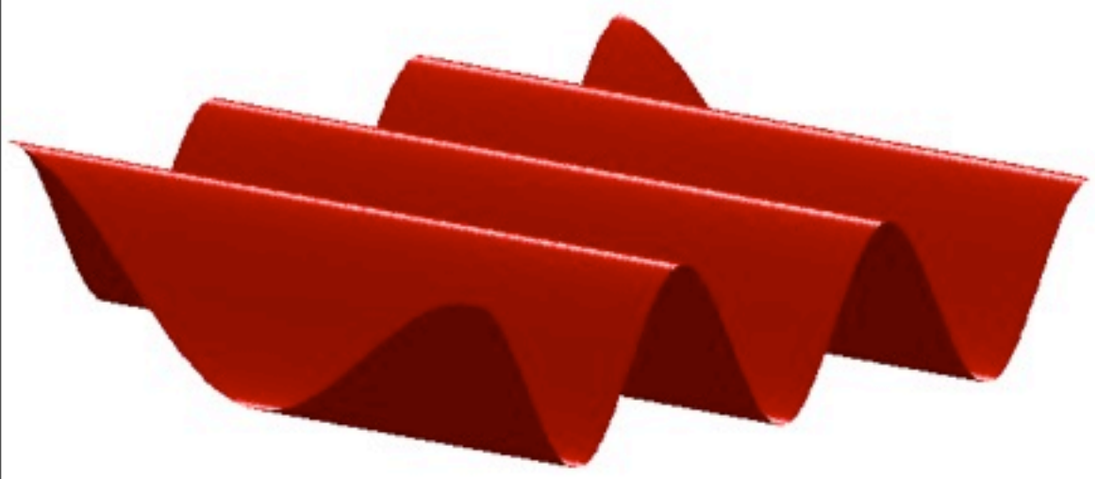
.... also circularly symmetric

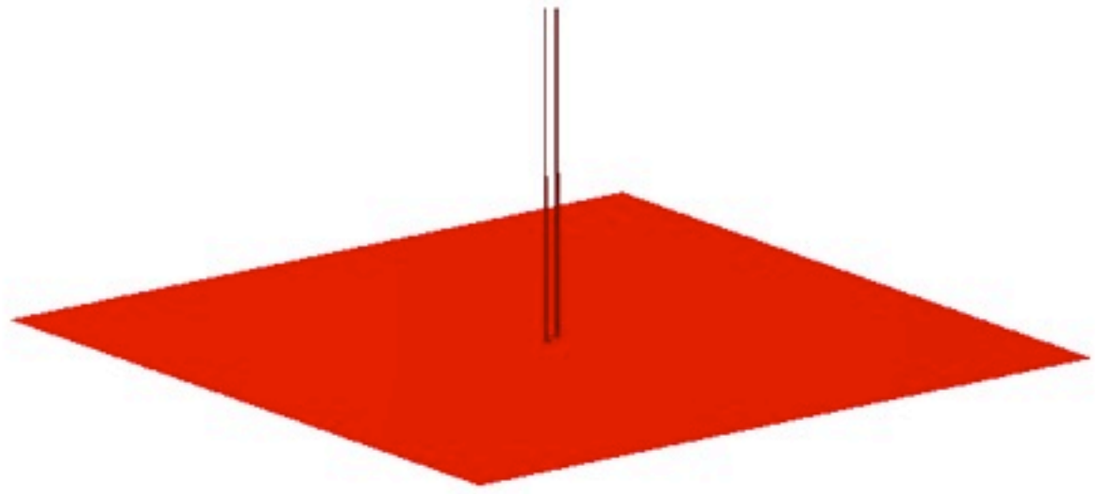
$$\Pi(r) \Longrightarrow \frac{J_1(\pi \rho_k)}{2\rho_k} \triangleq \mathrm{jinc}(\rho_k)$$

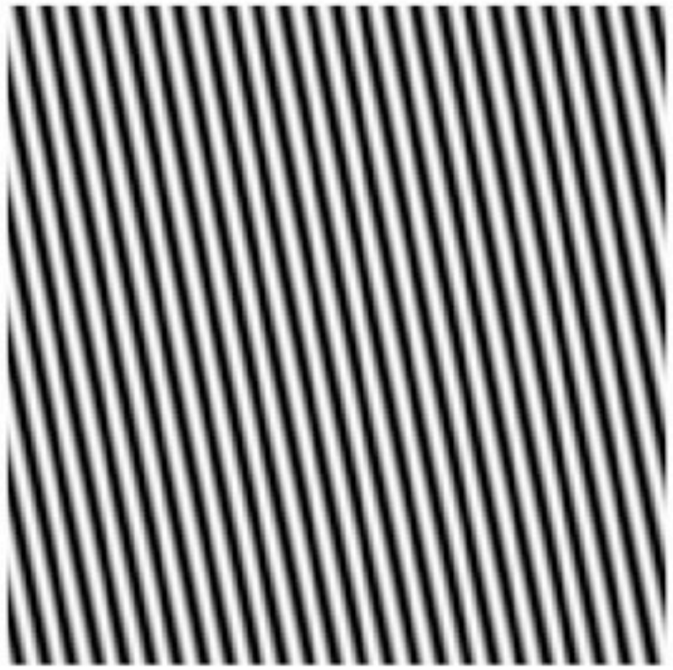
like a sinc — but different

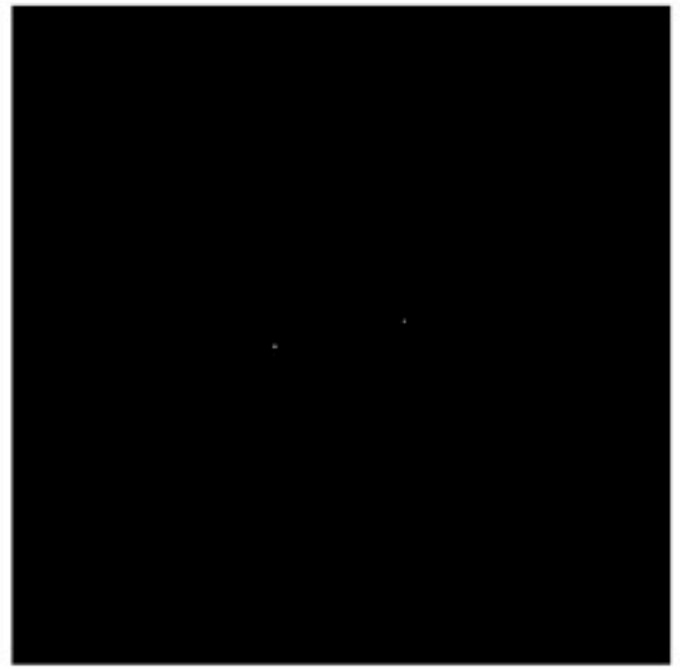


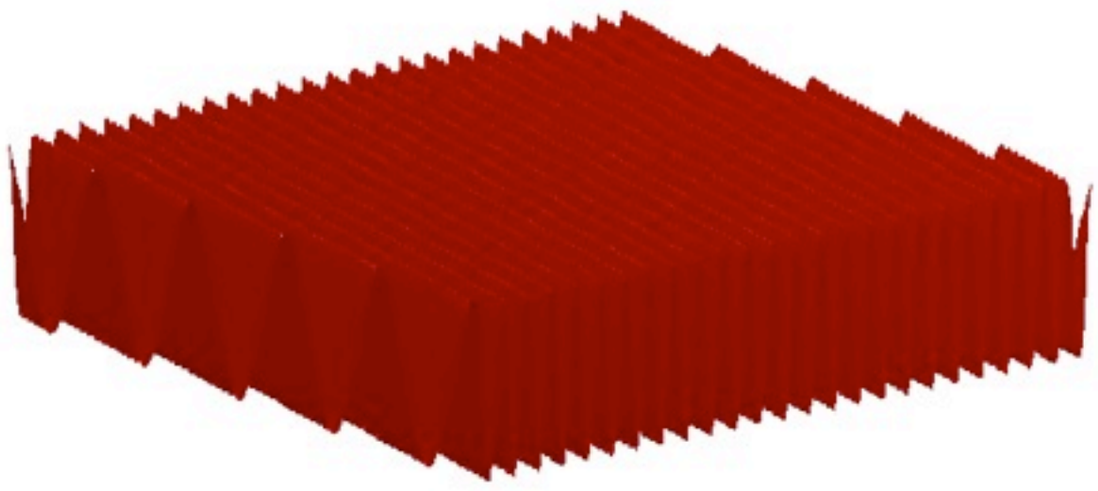


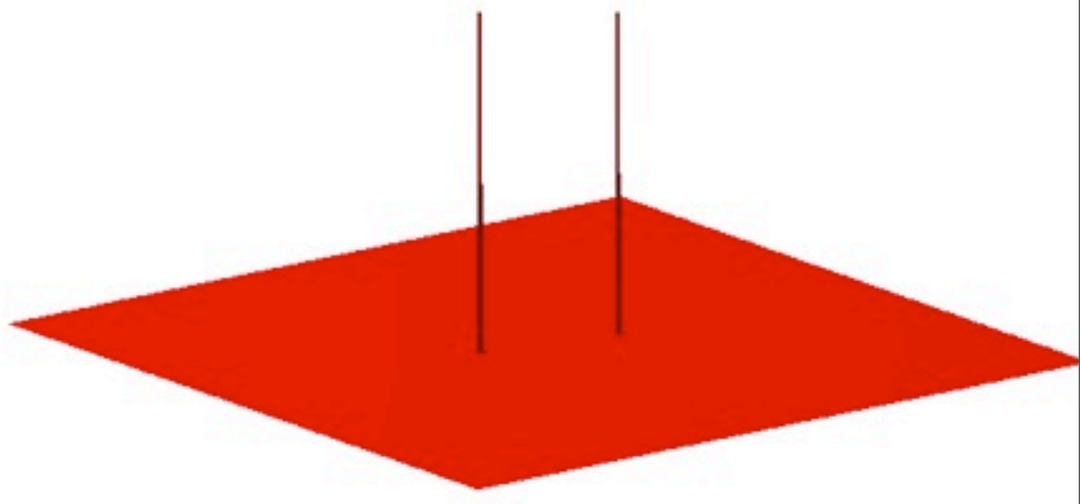


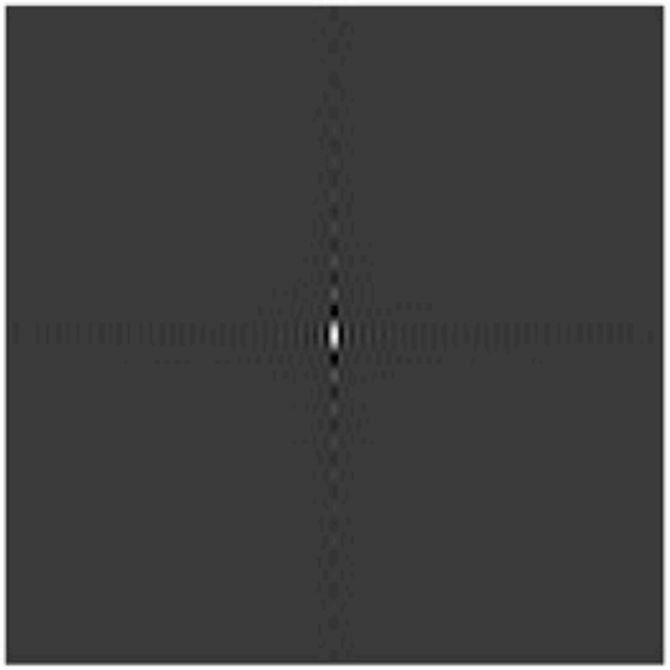


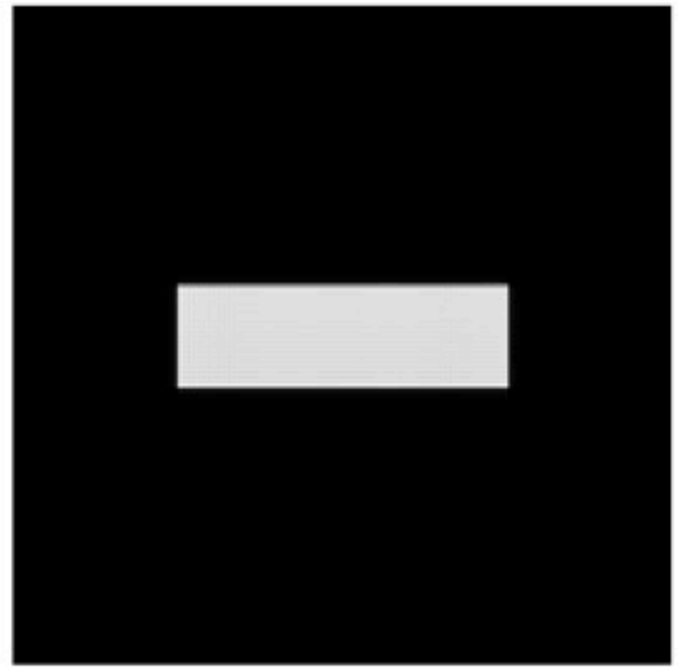


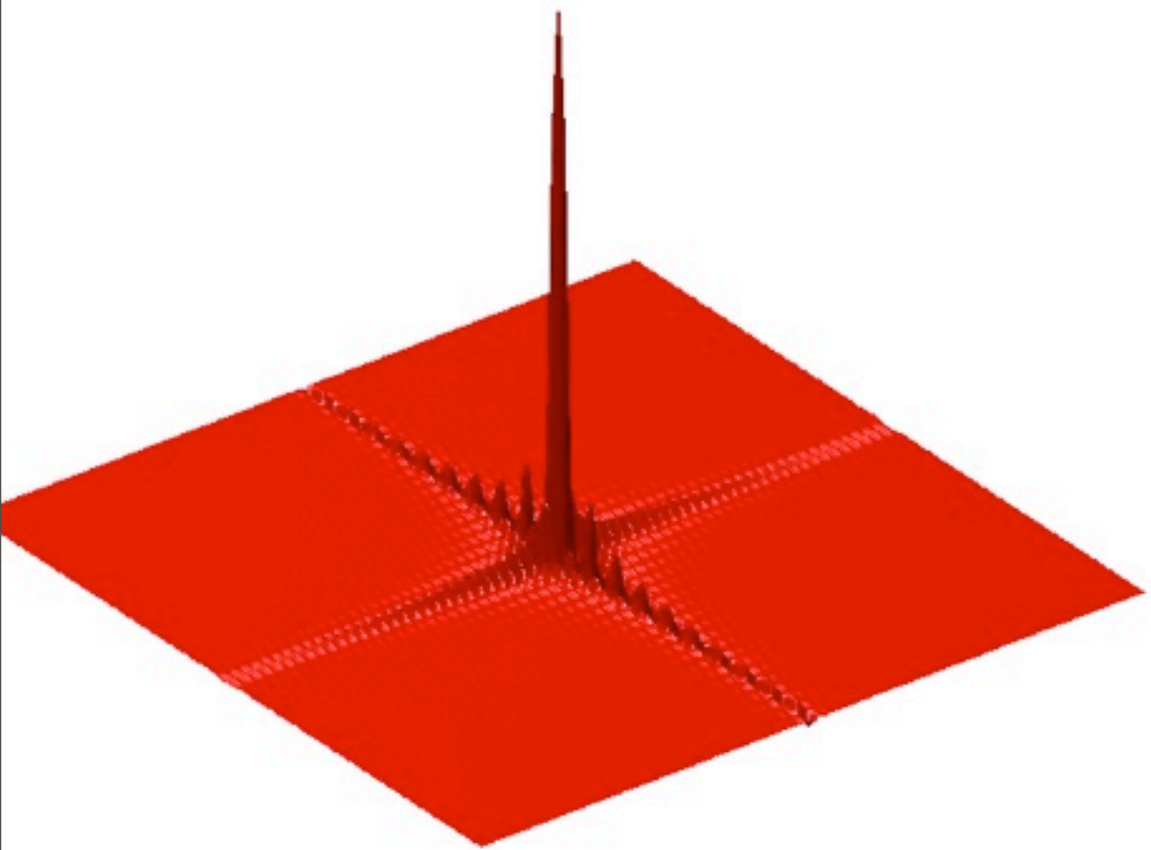


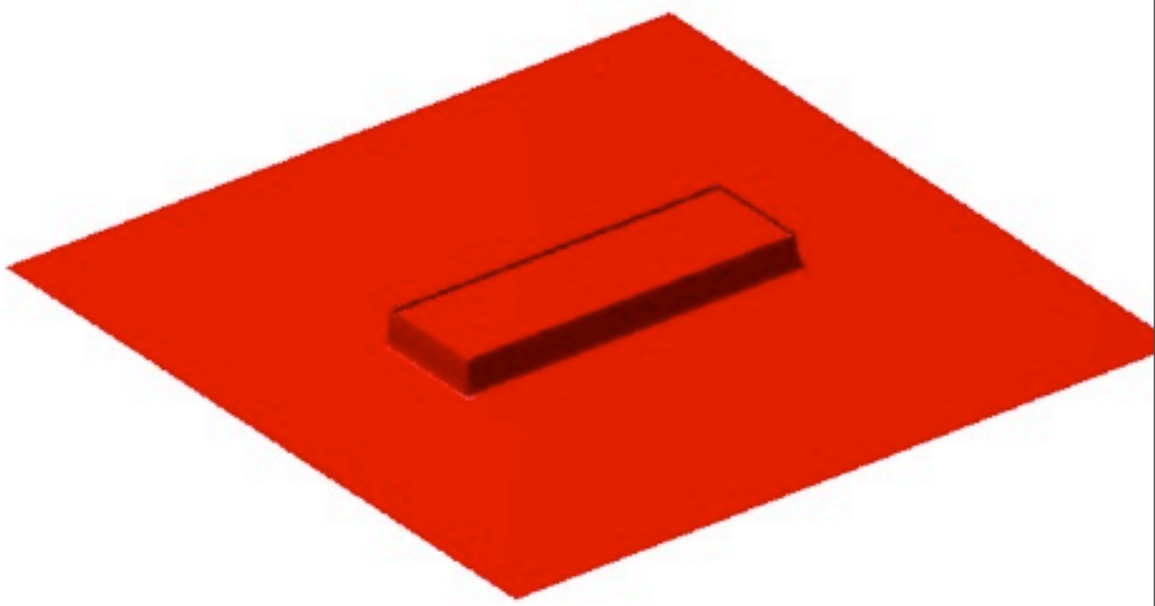


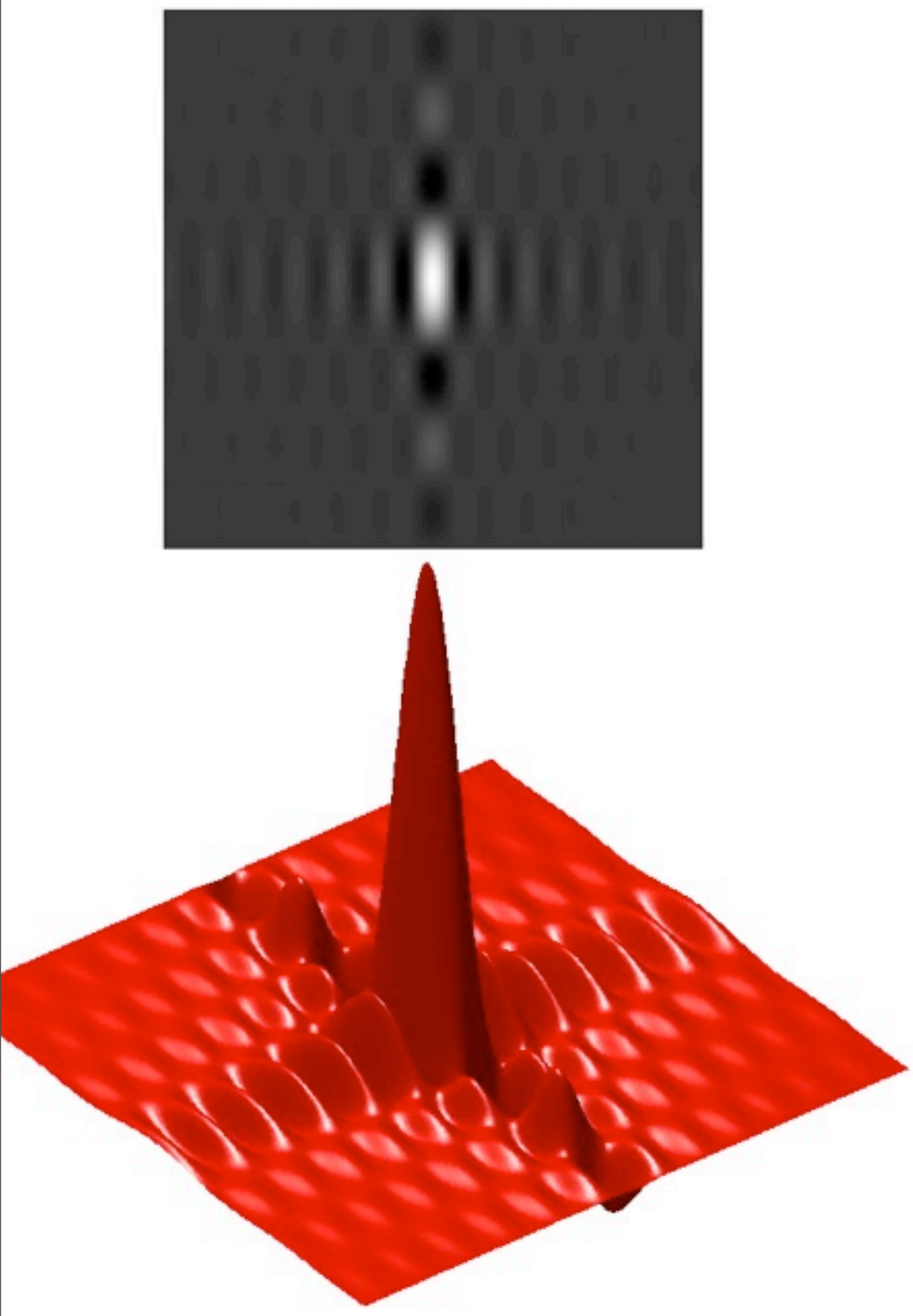


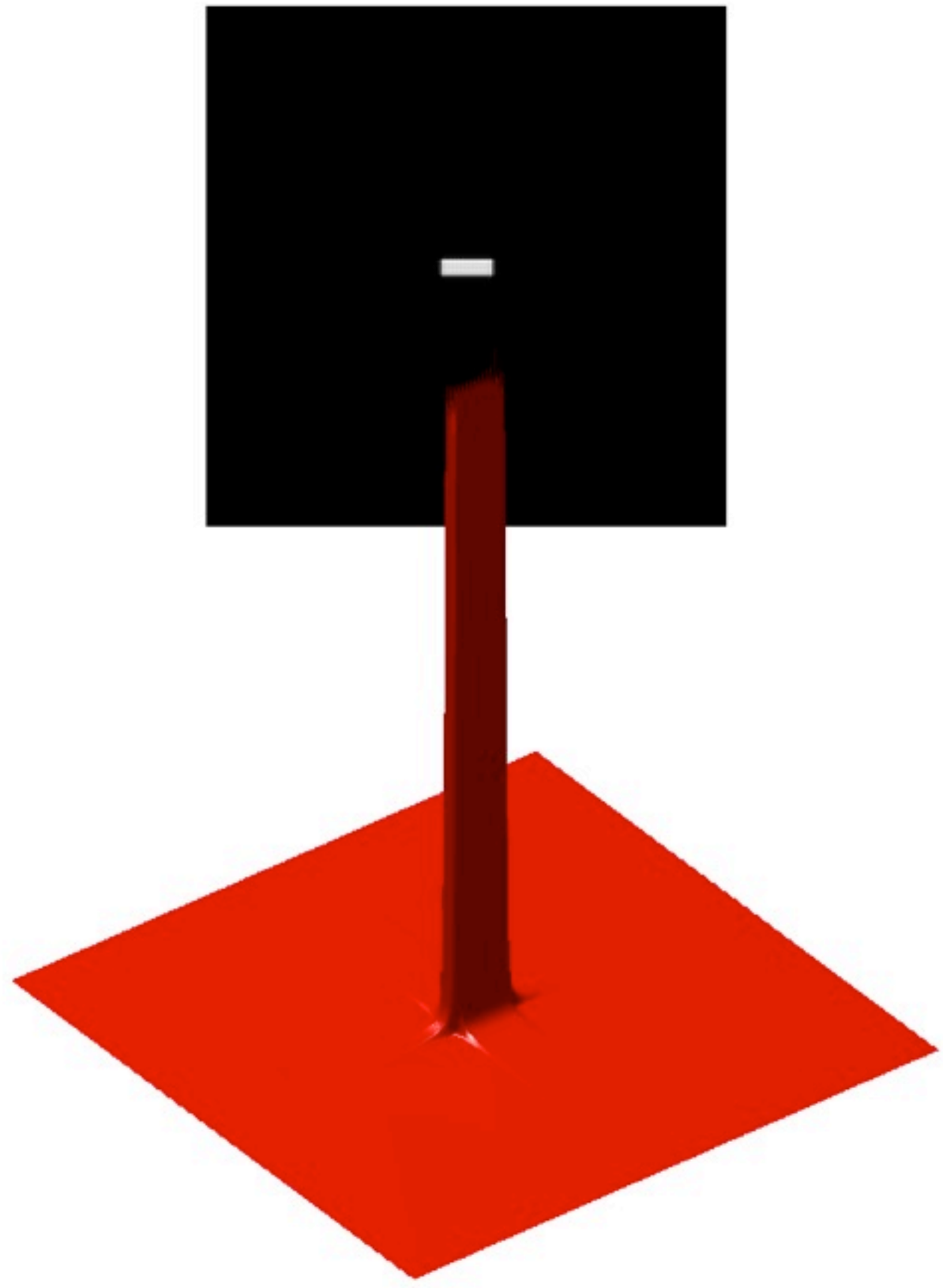


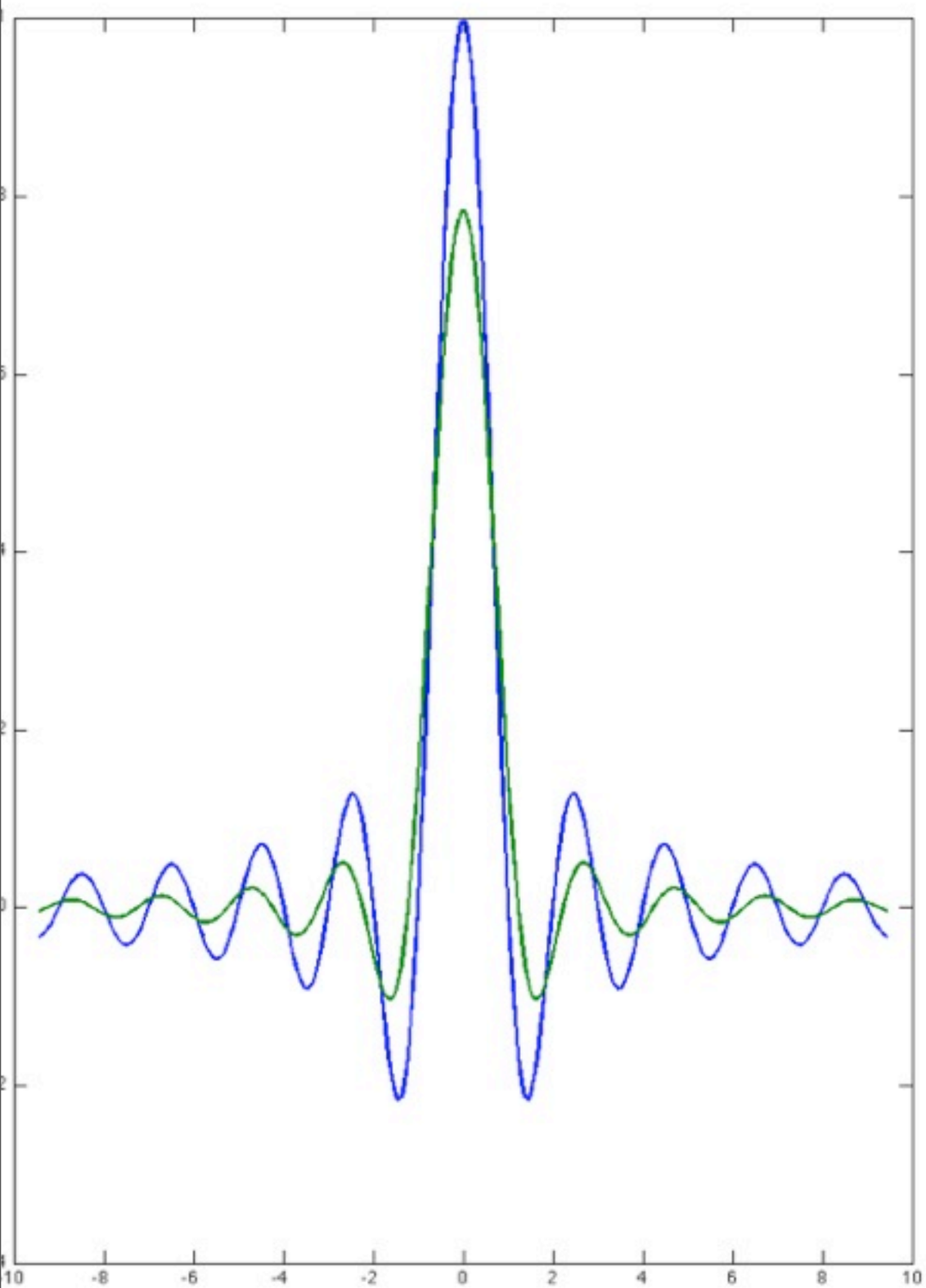




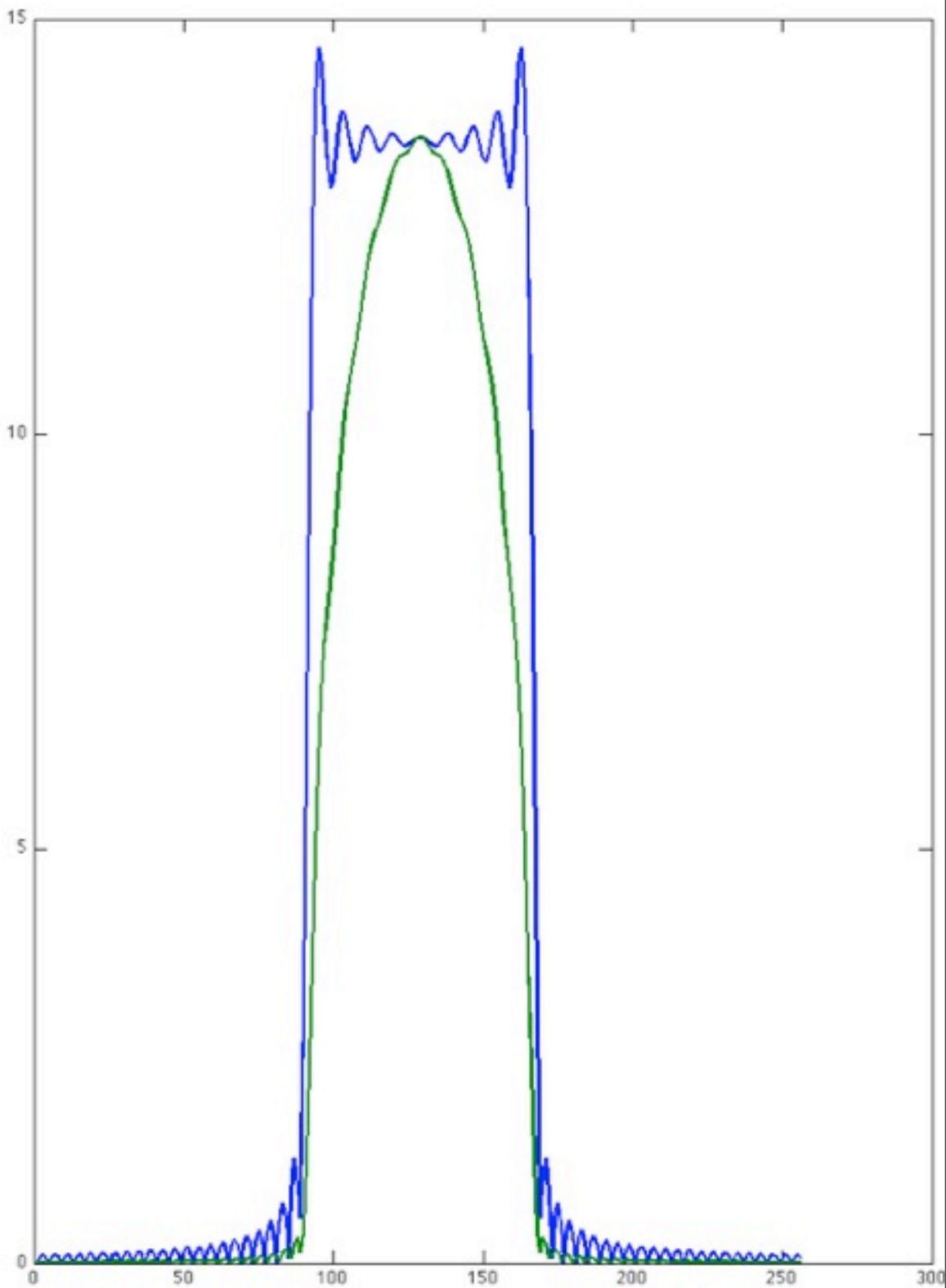


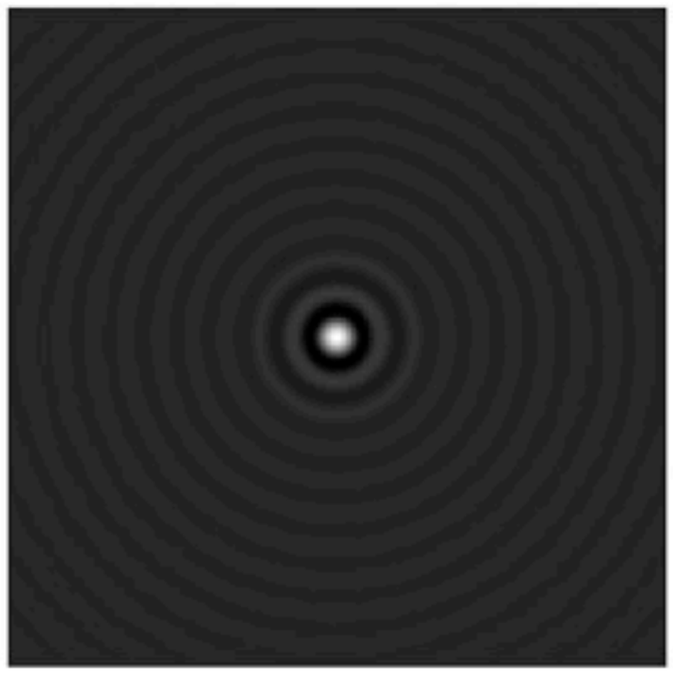


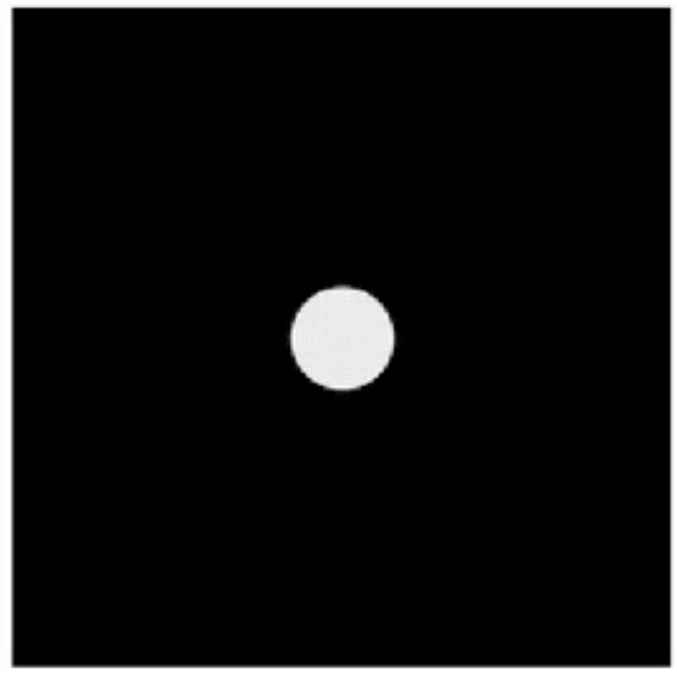


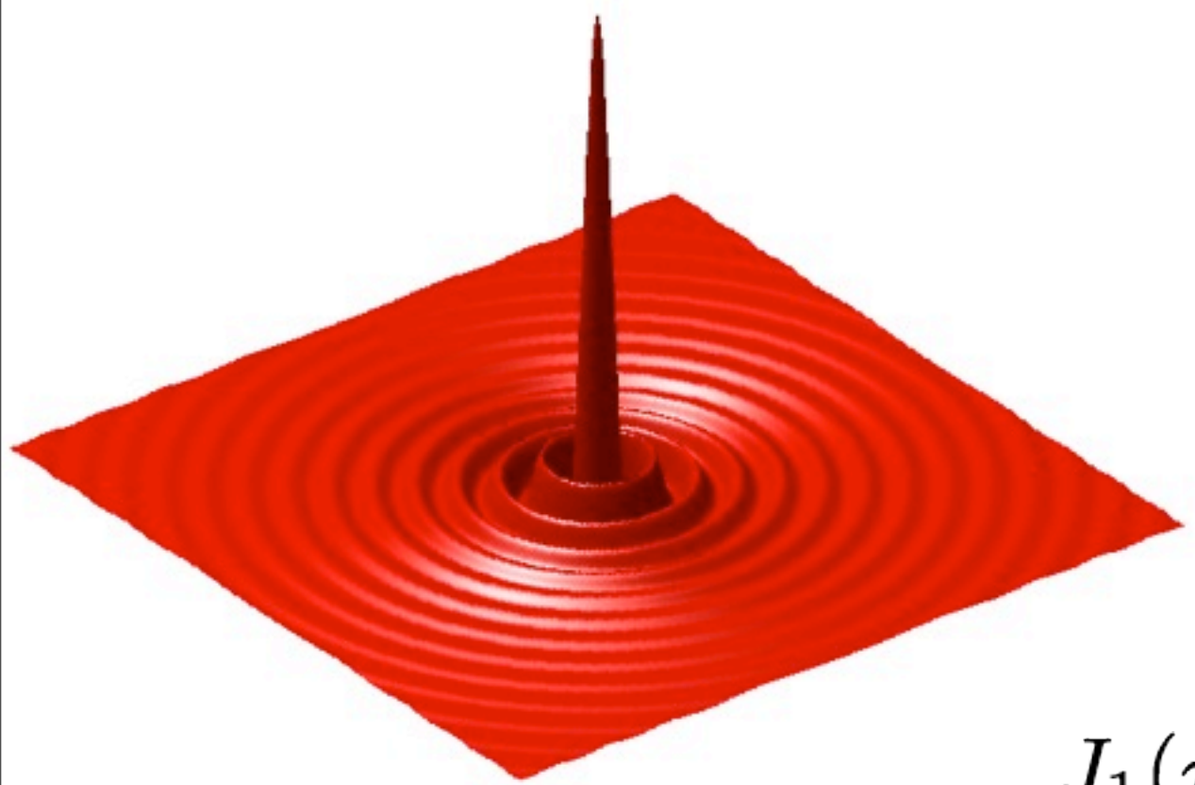


$$\frac{J_1(\pi x)}{2x} \qquad \frac{\sin(\pi x)}{\pi x}$$

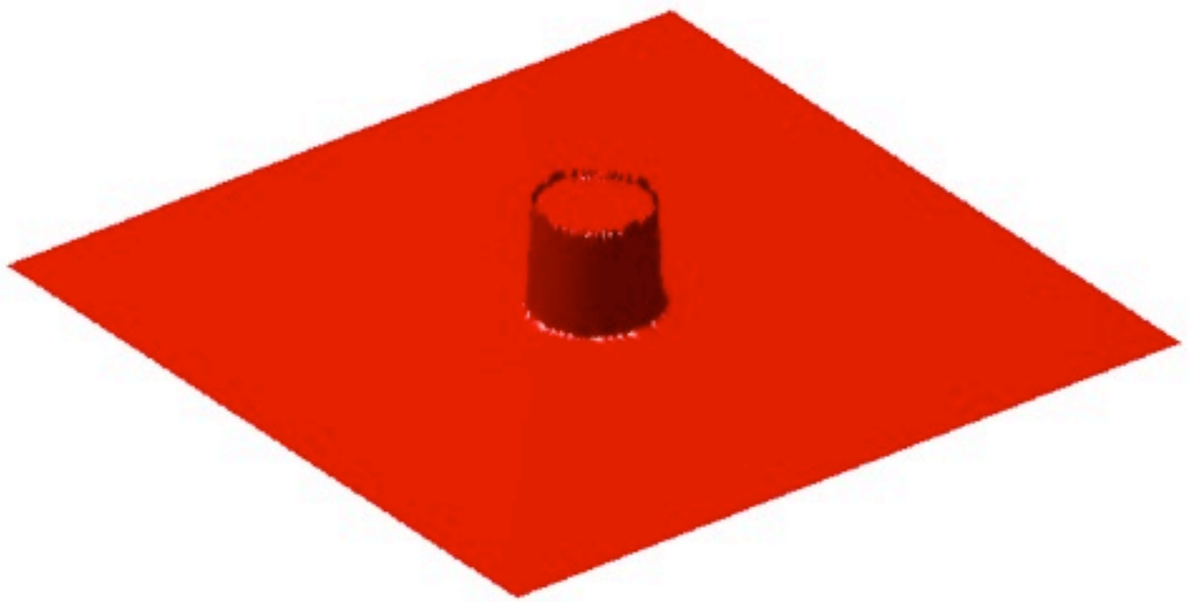


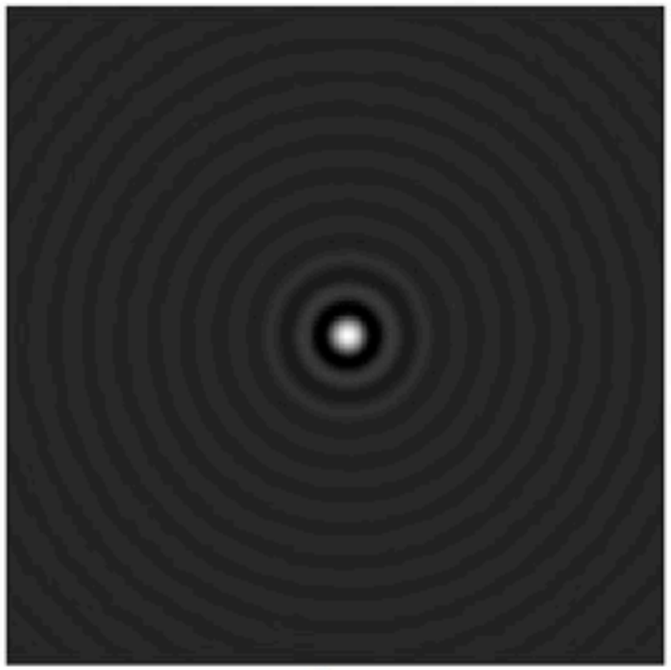


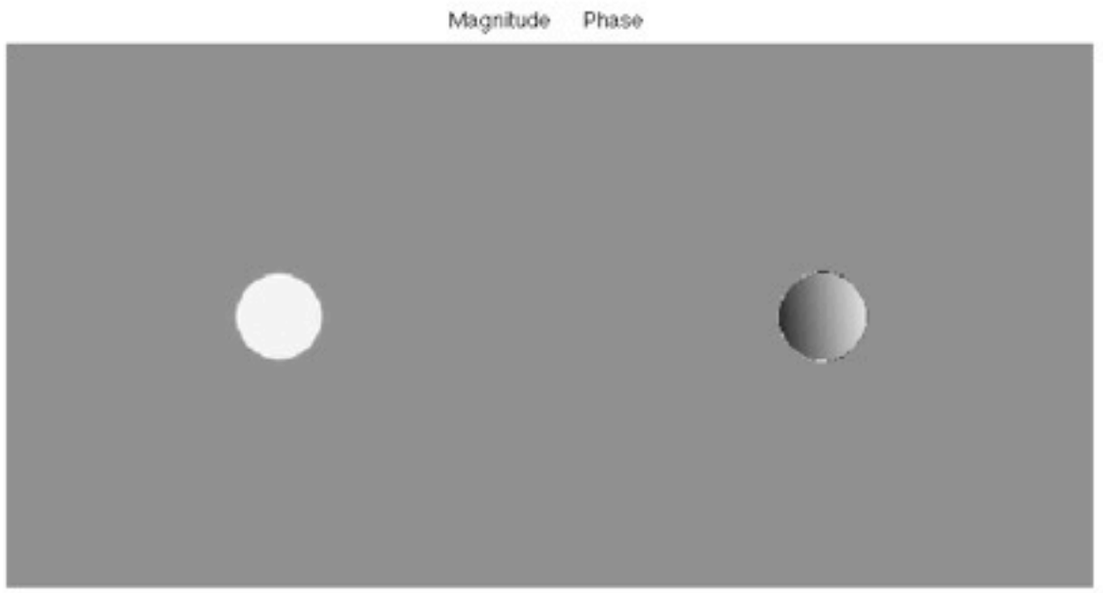
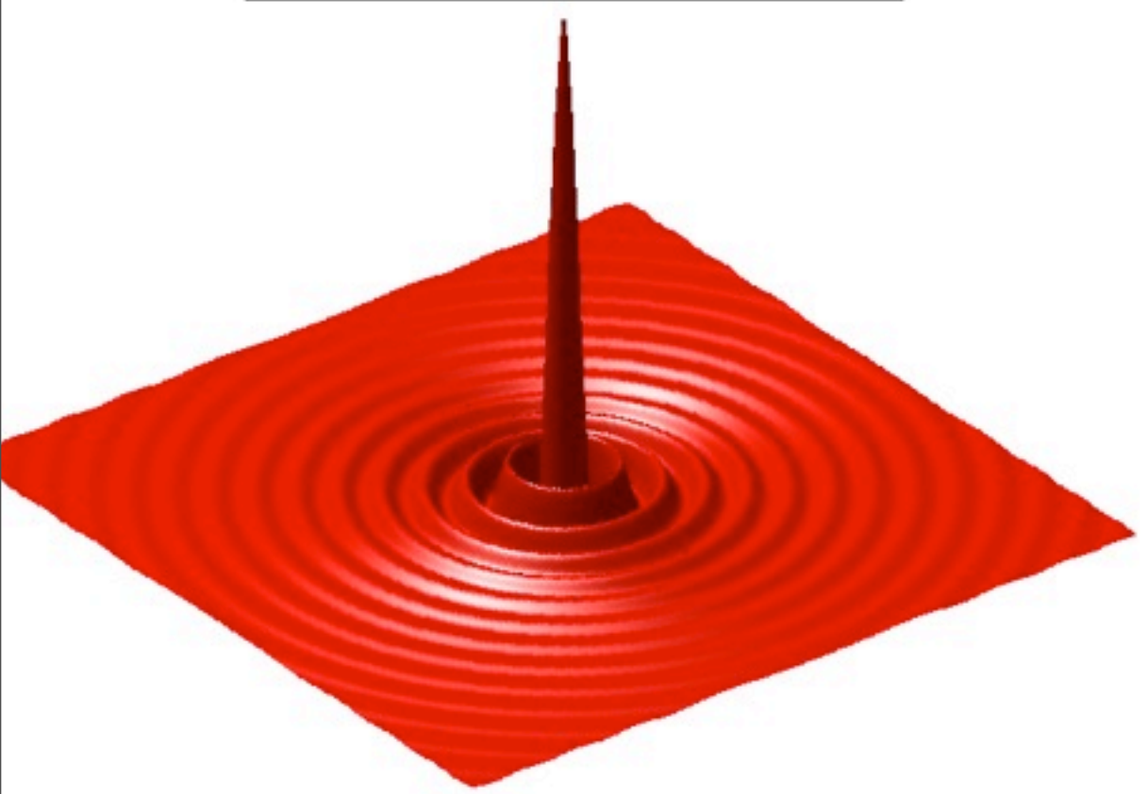
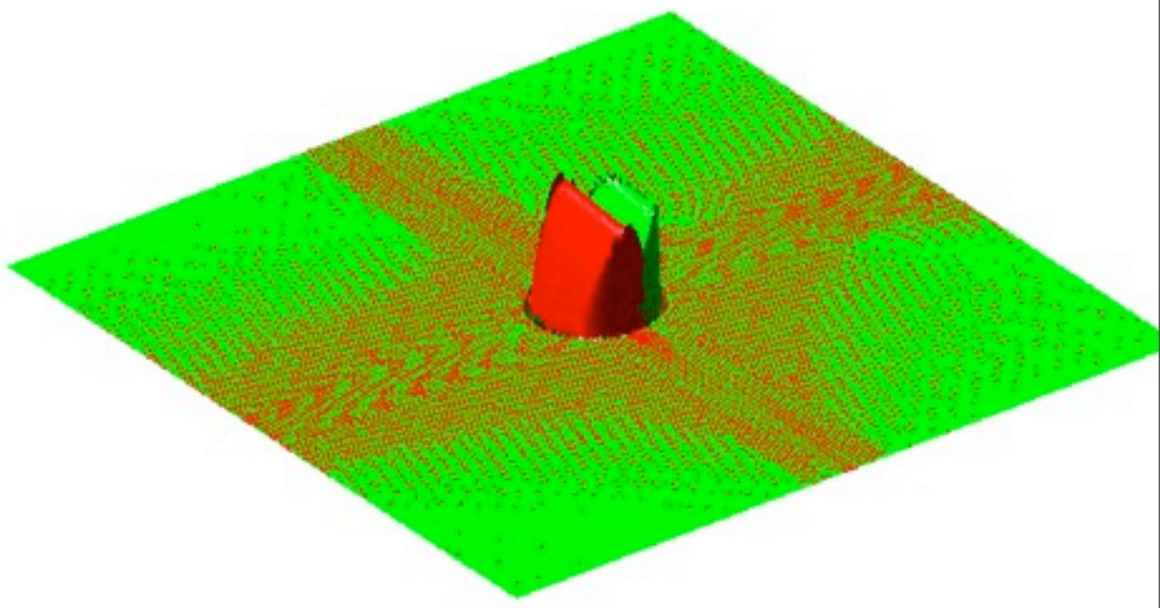
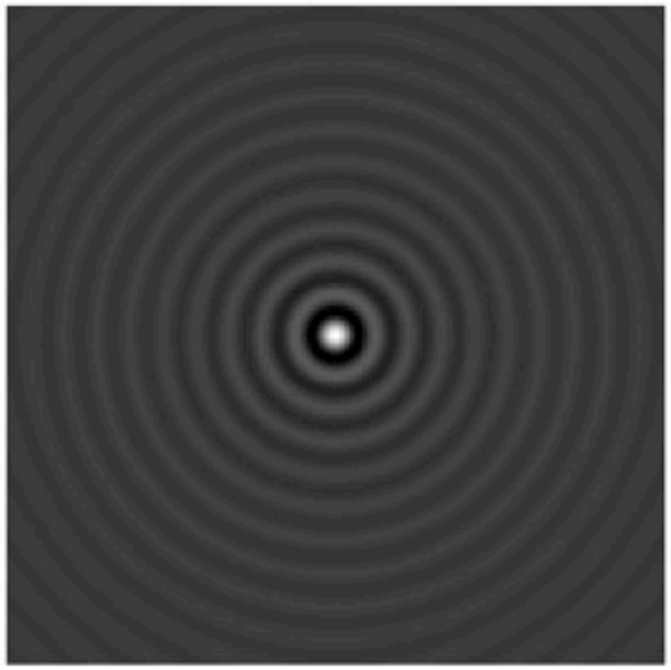
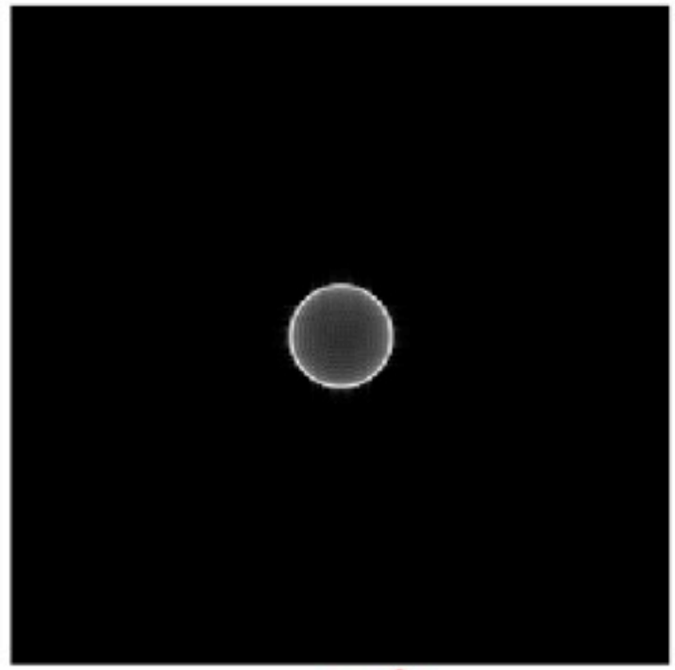
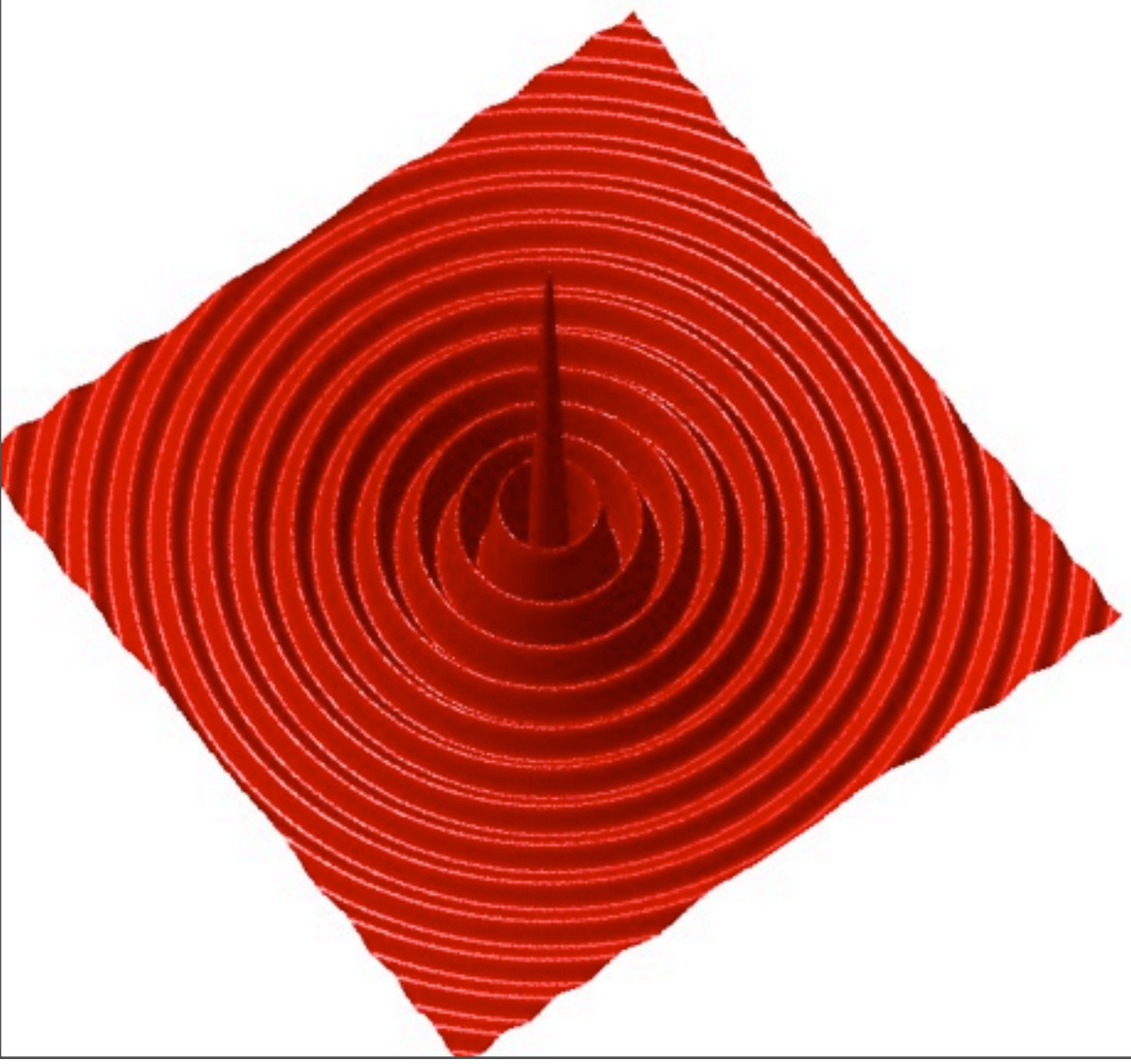
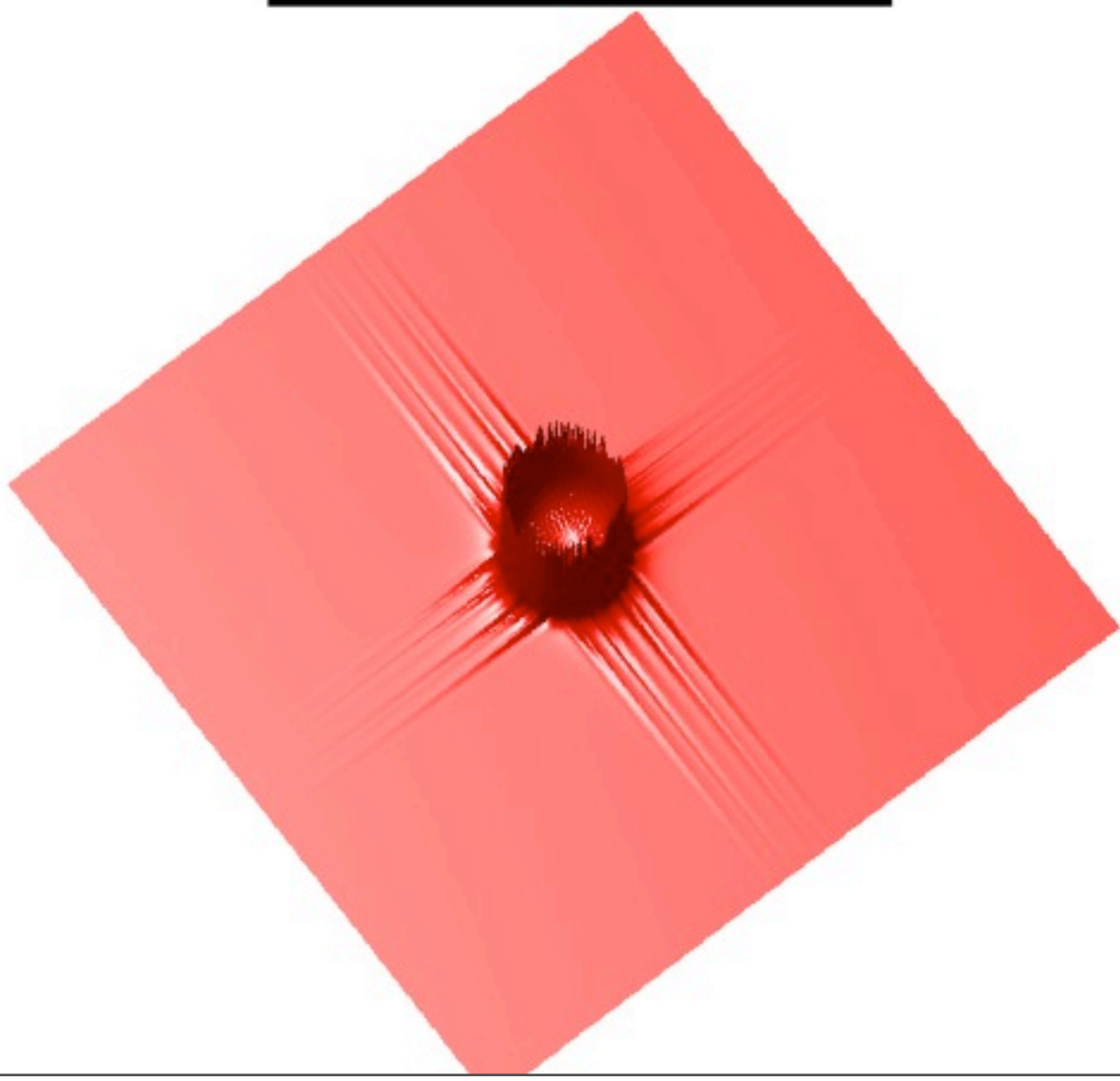
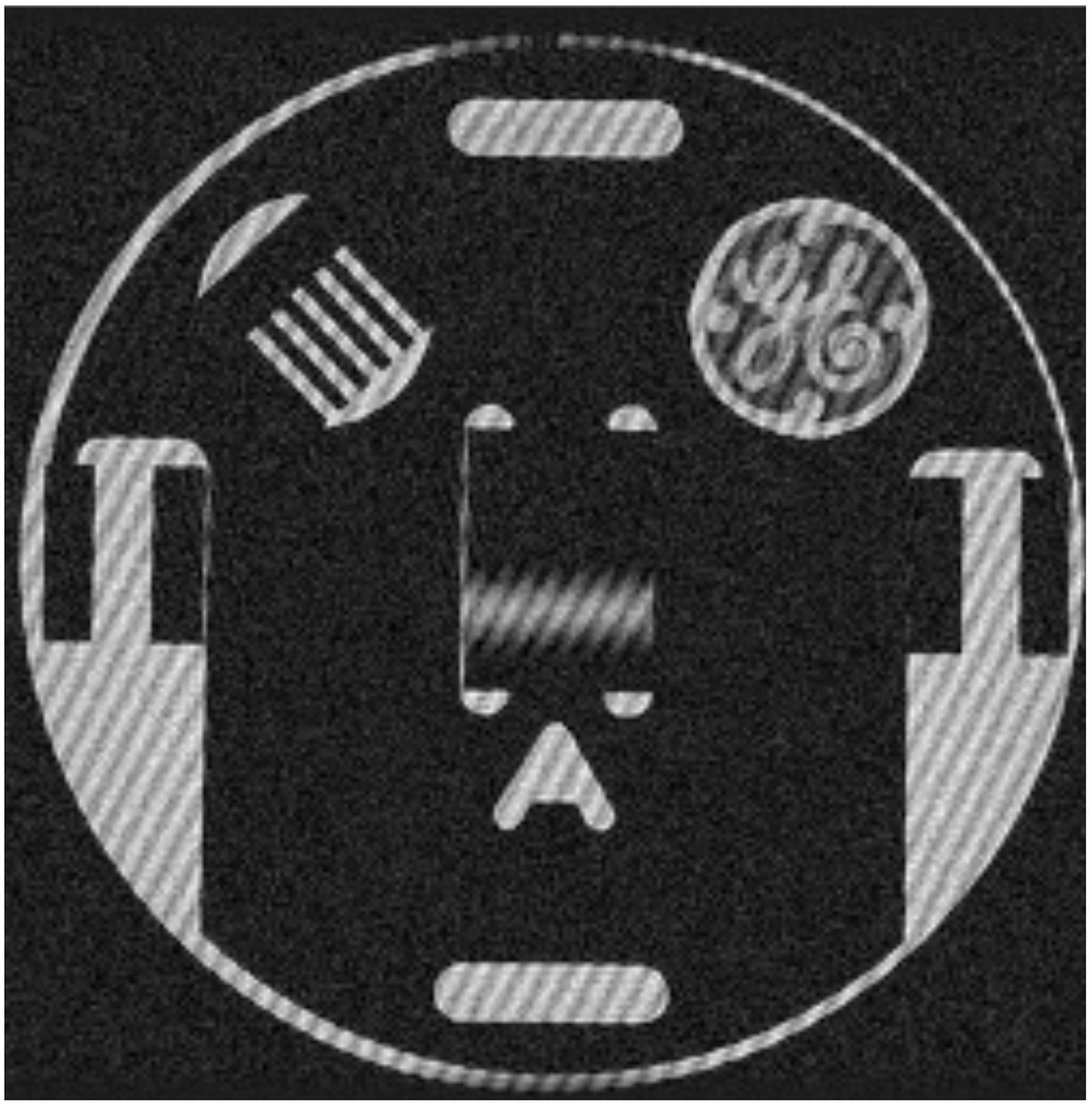
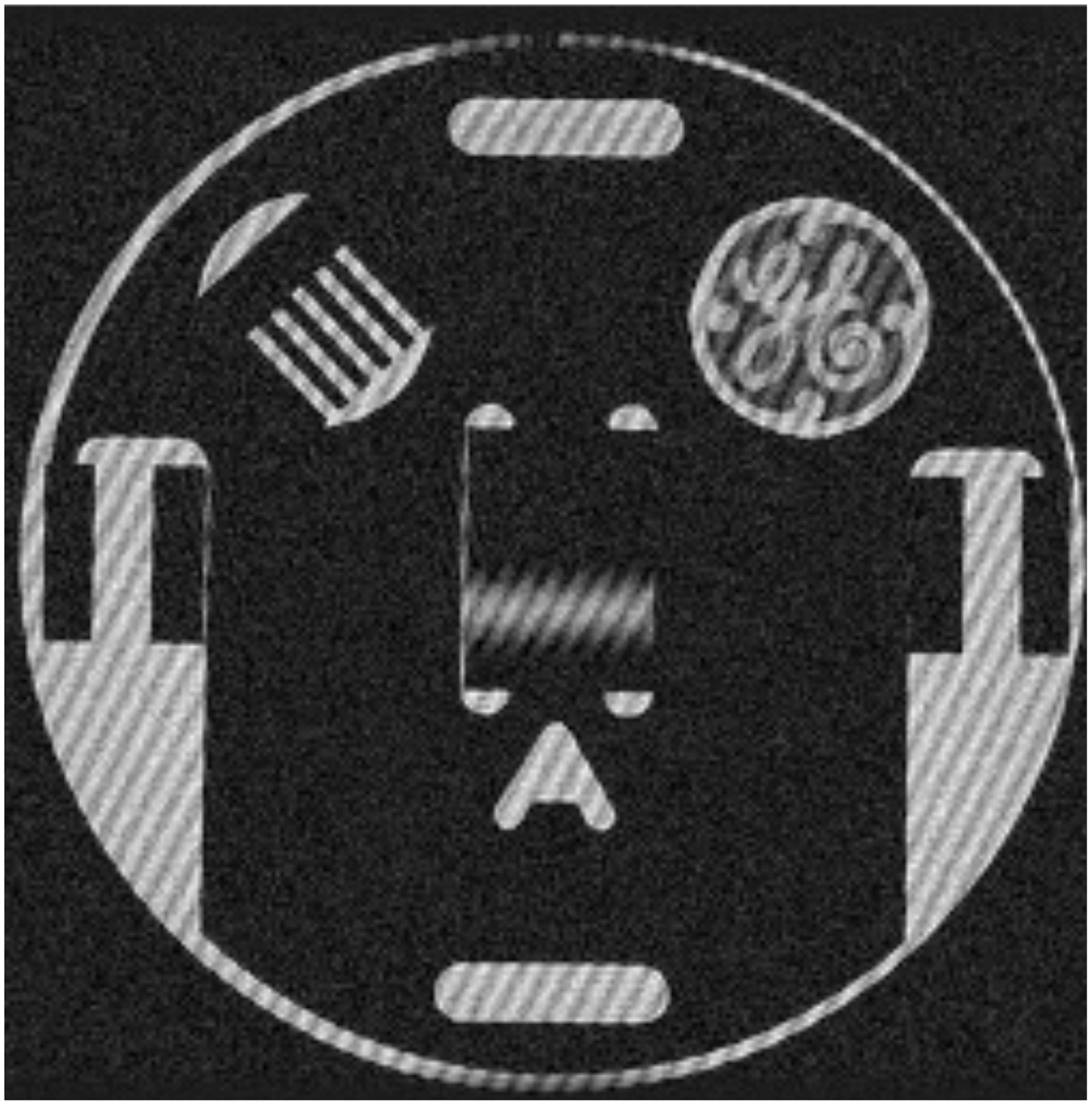
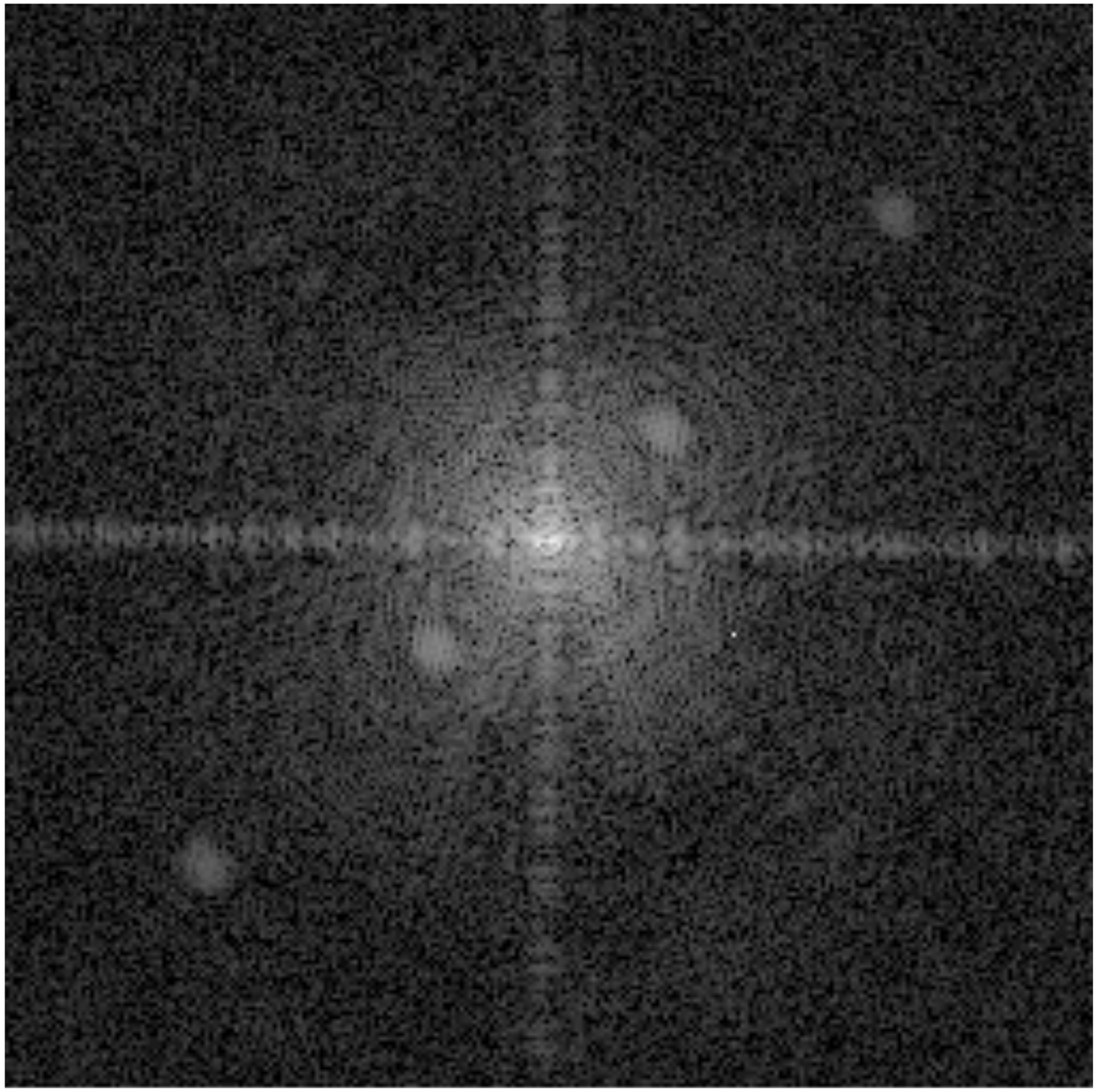
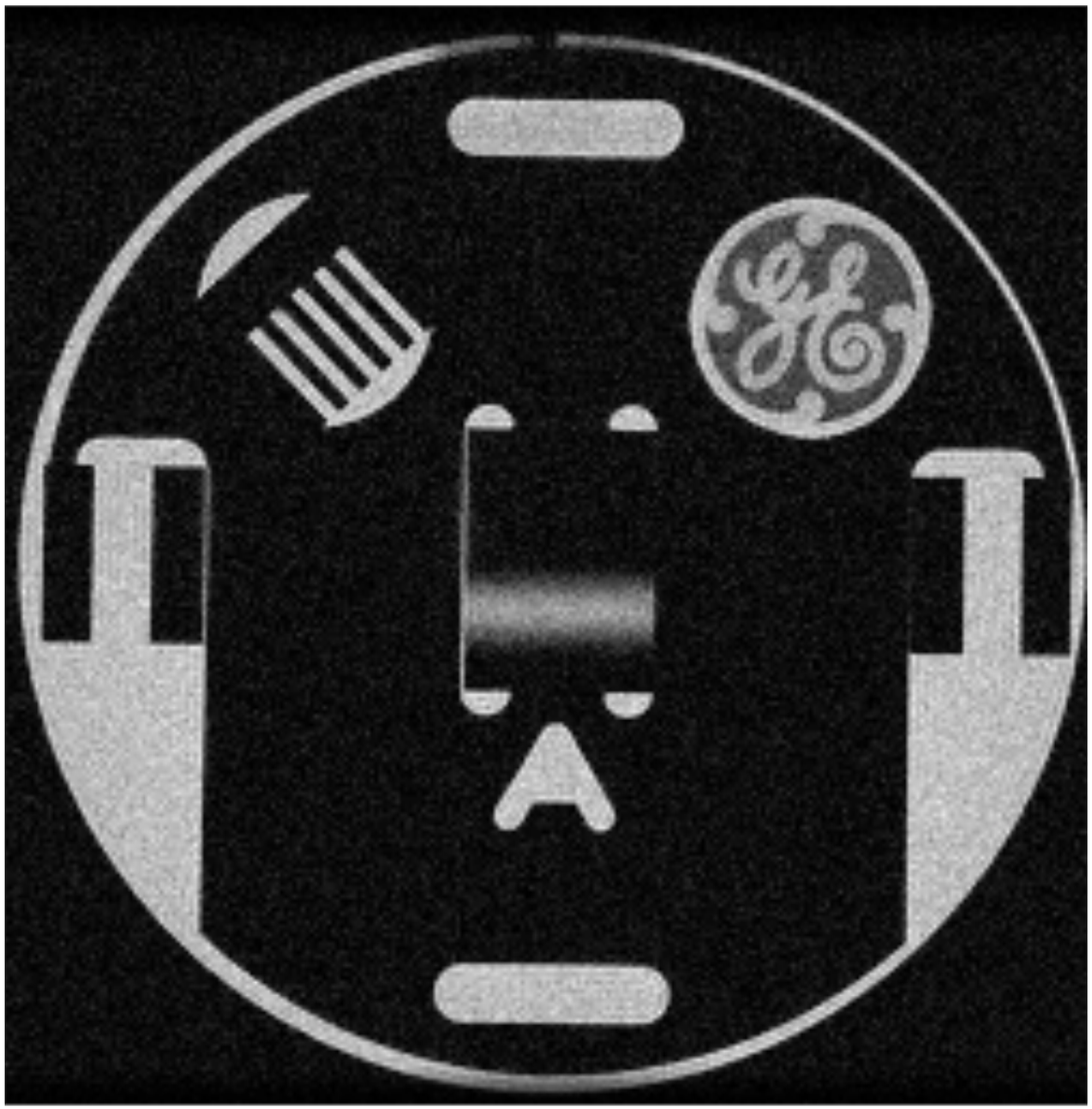
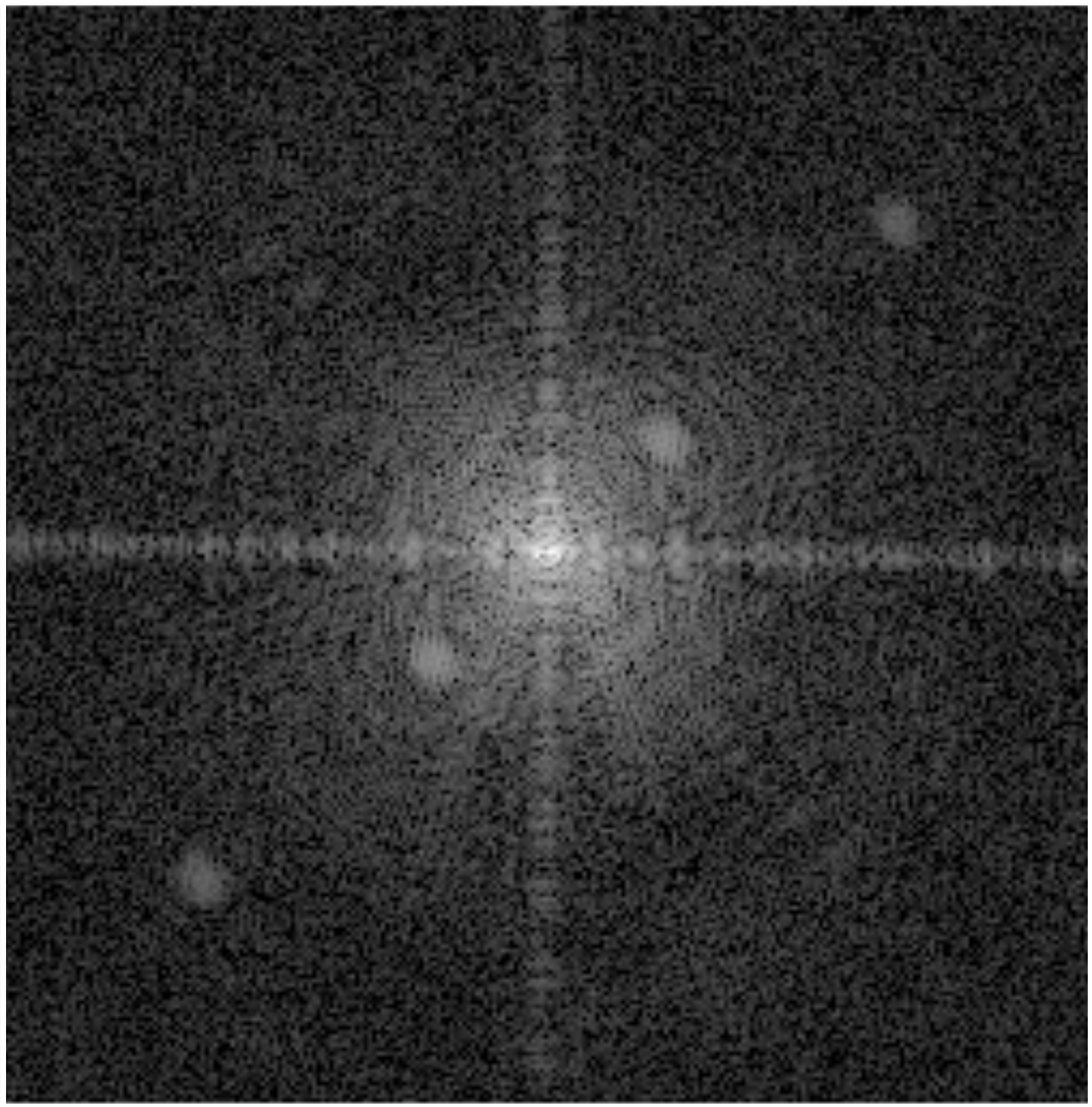
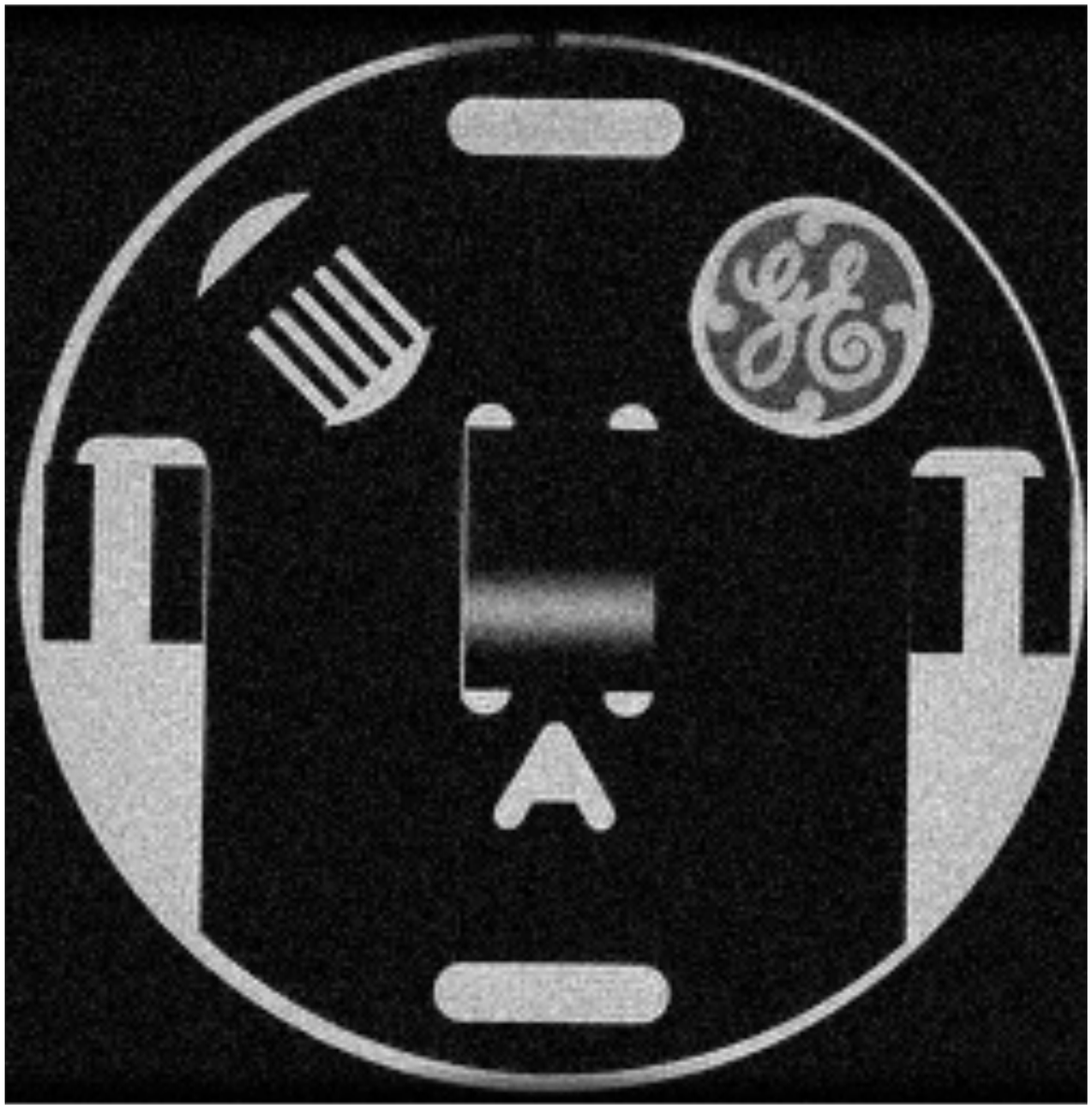
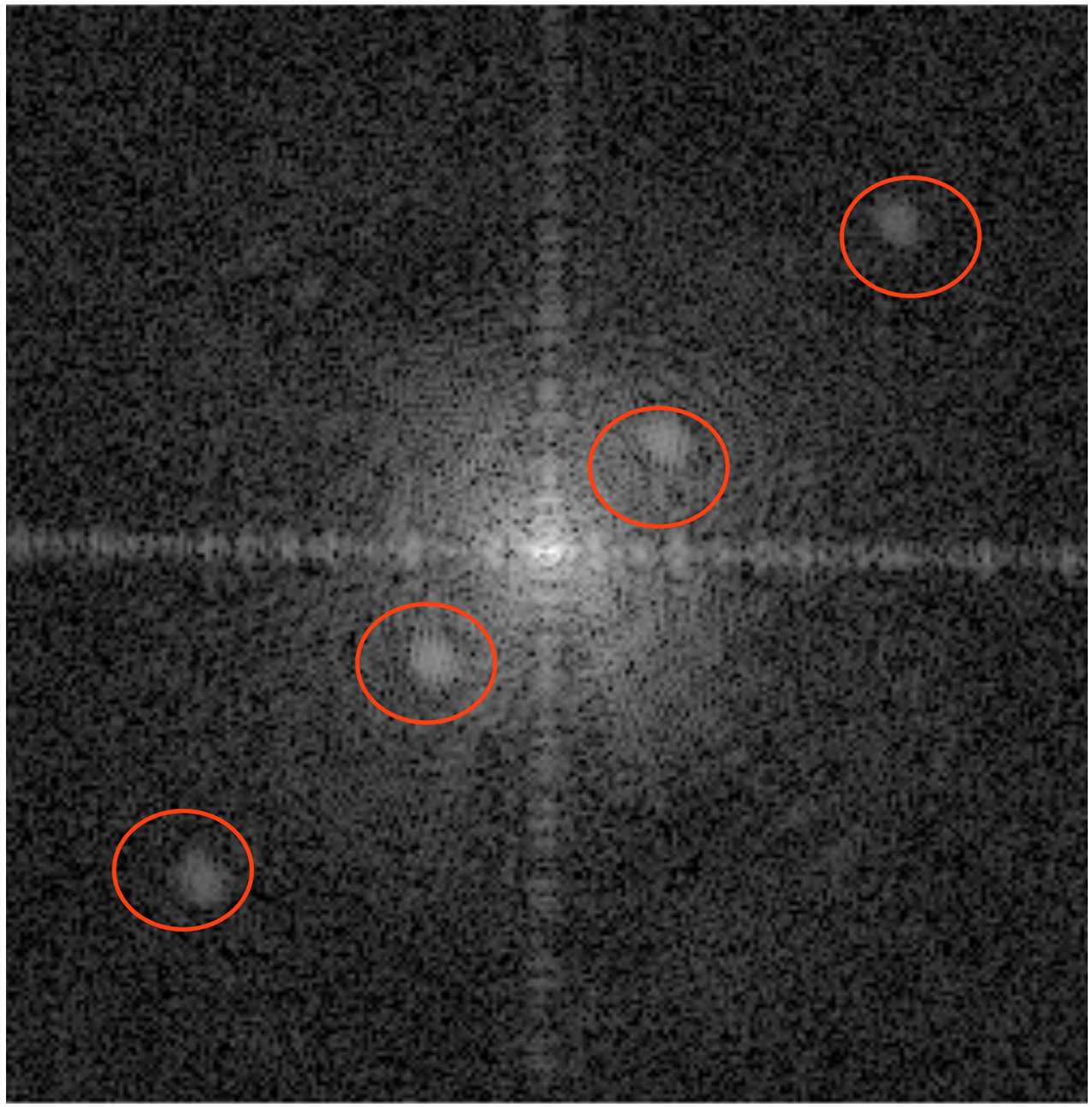
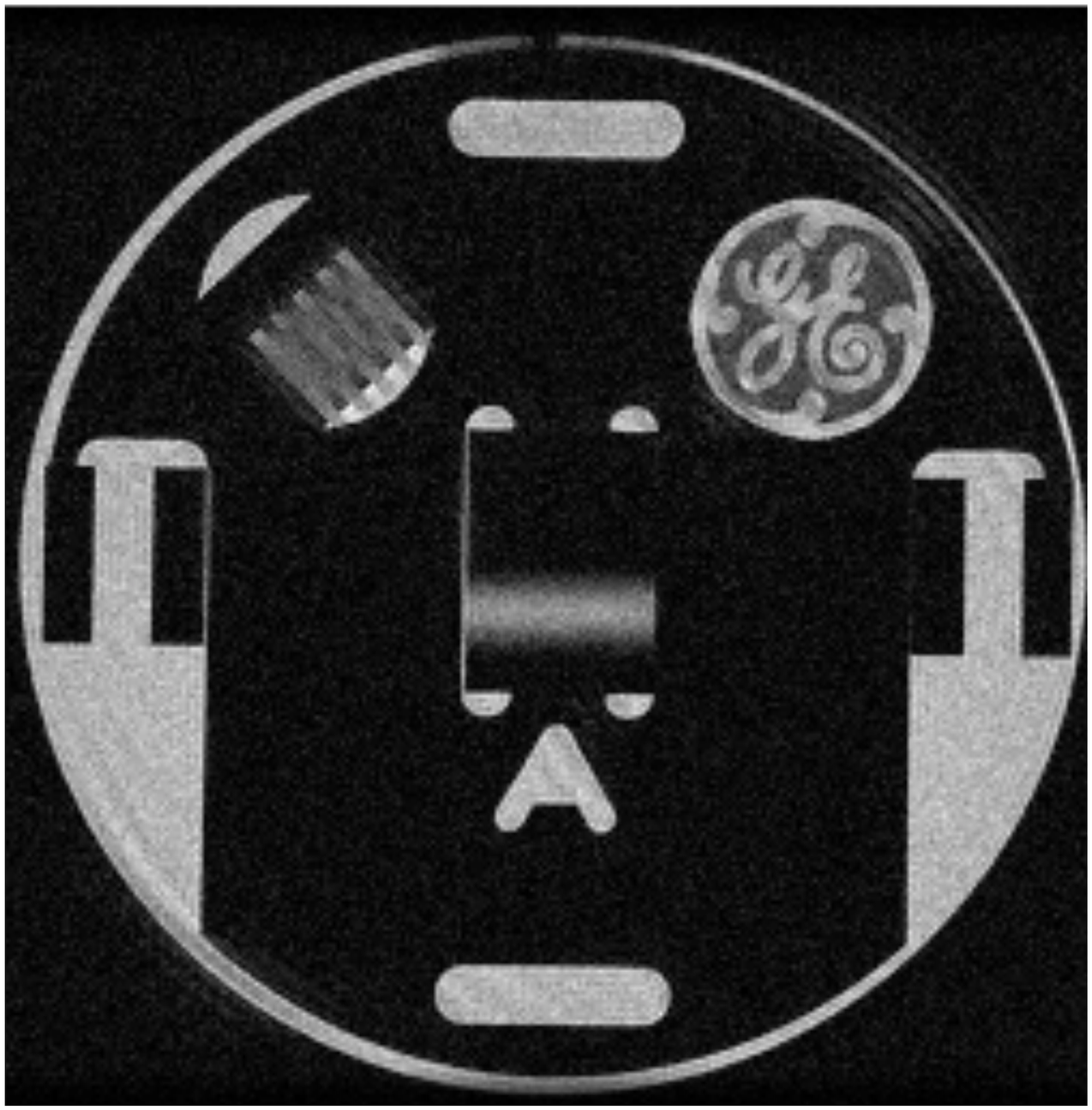
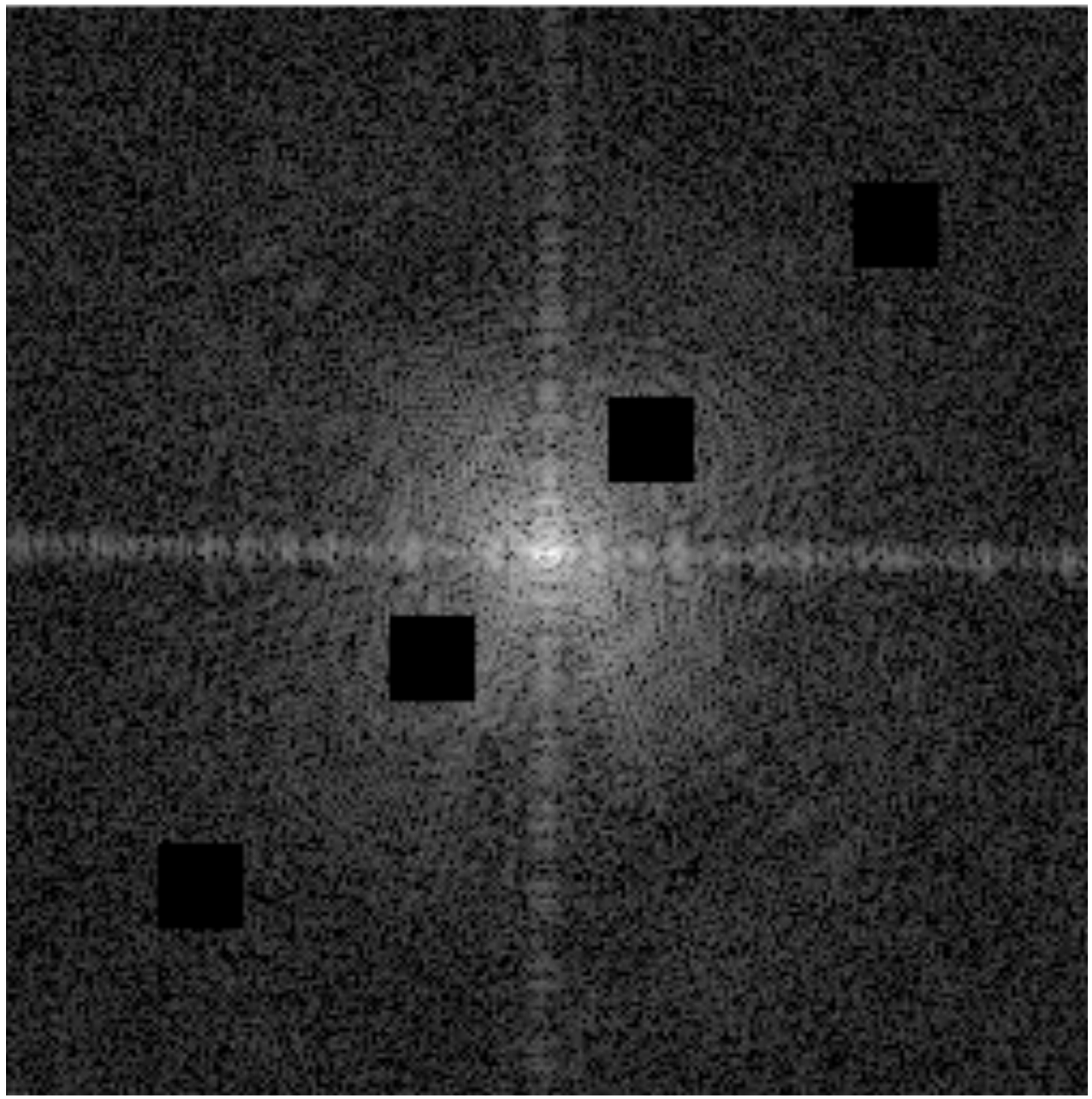
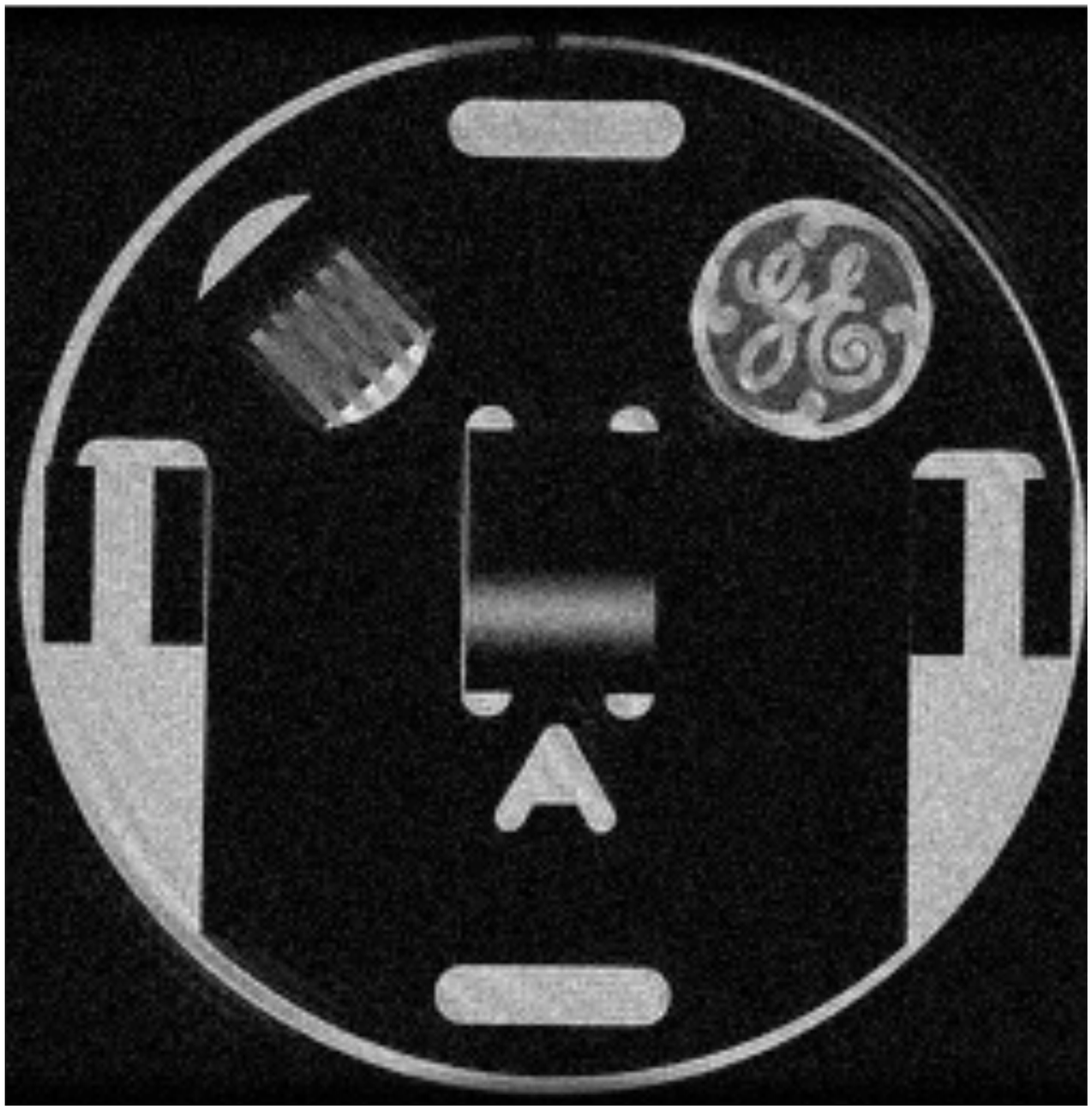
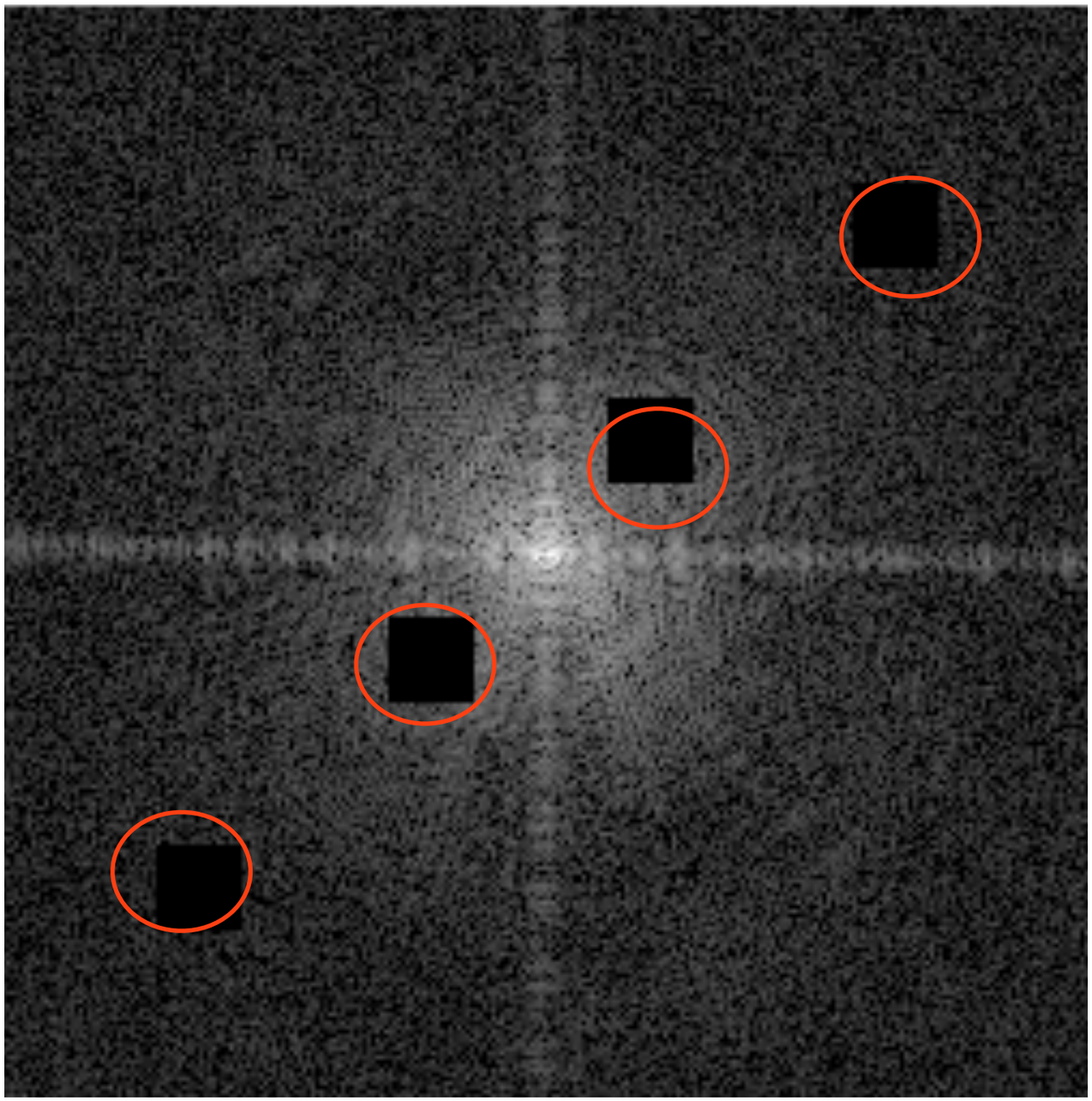
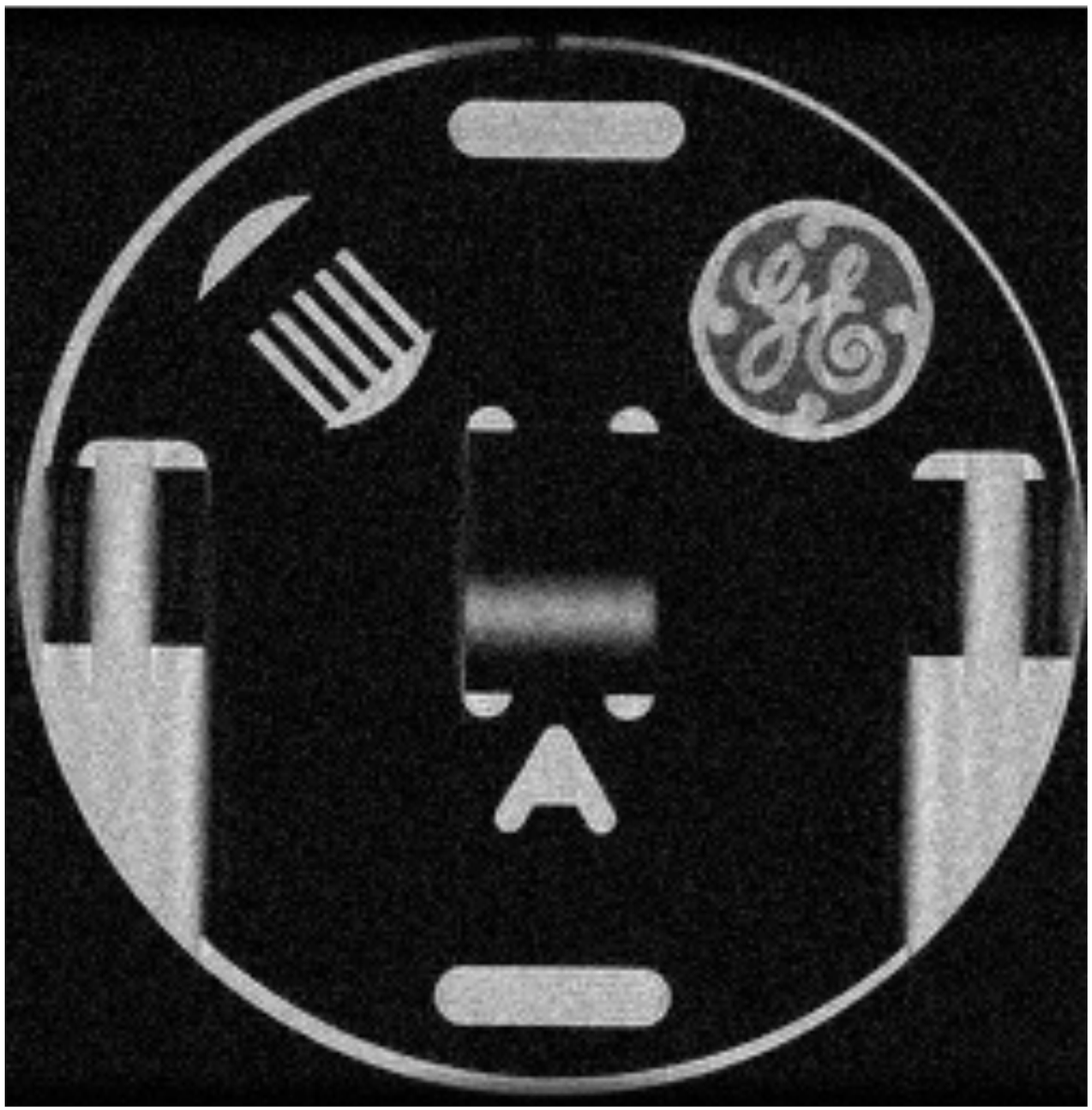
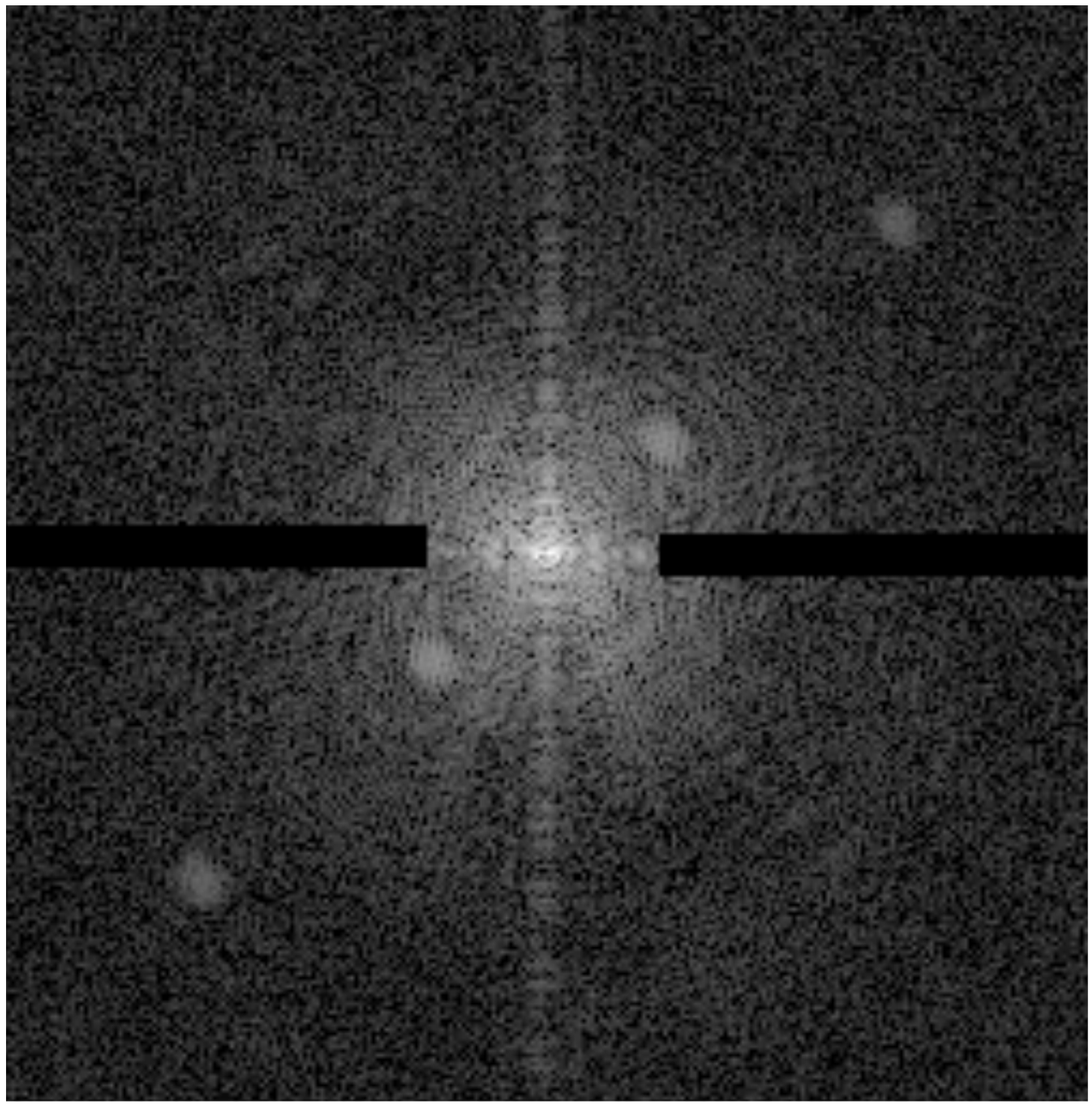
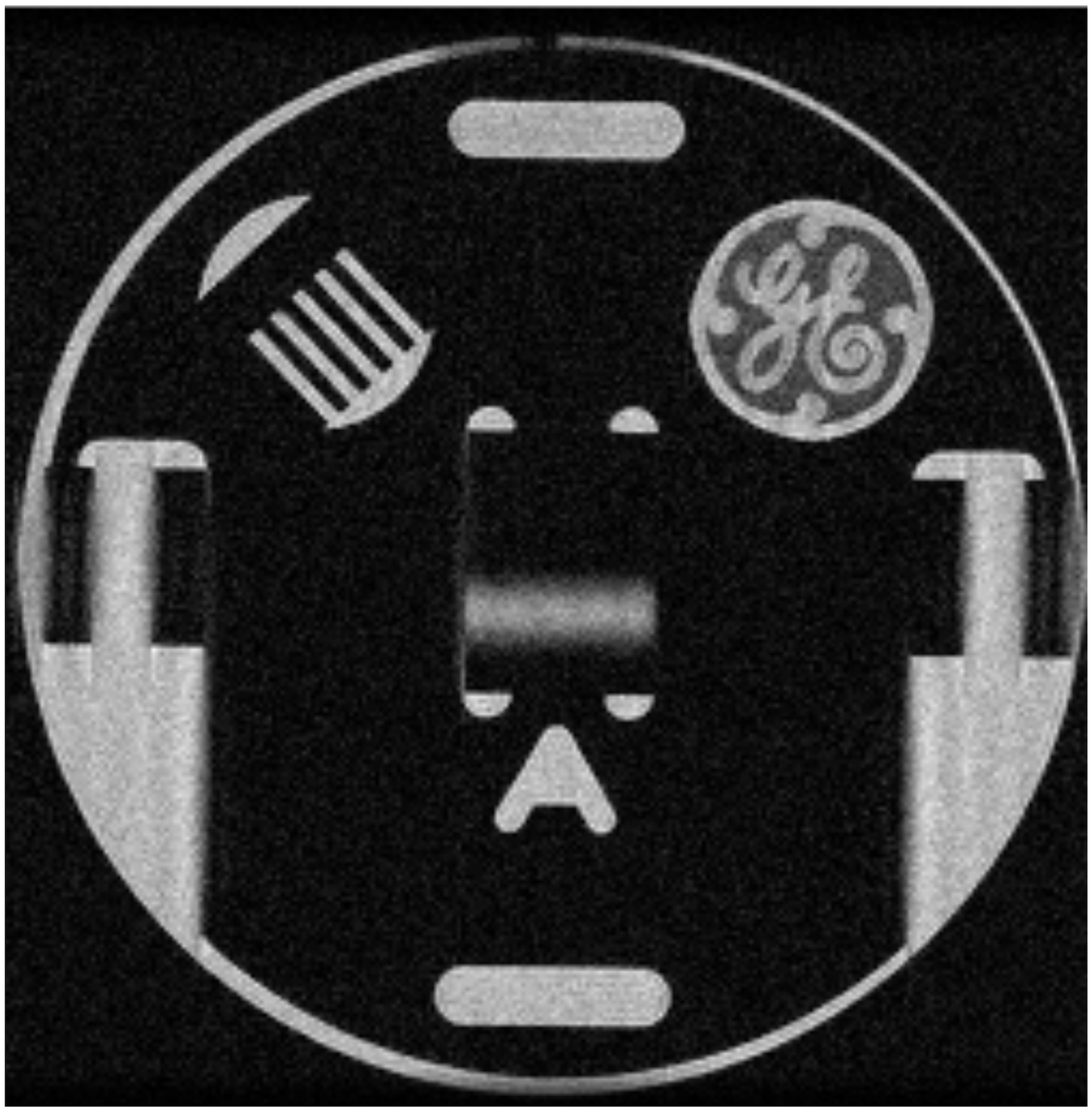
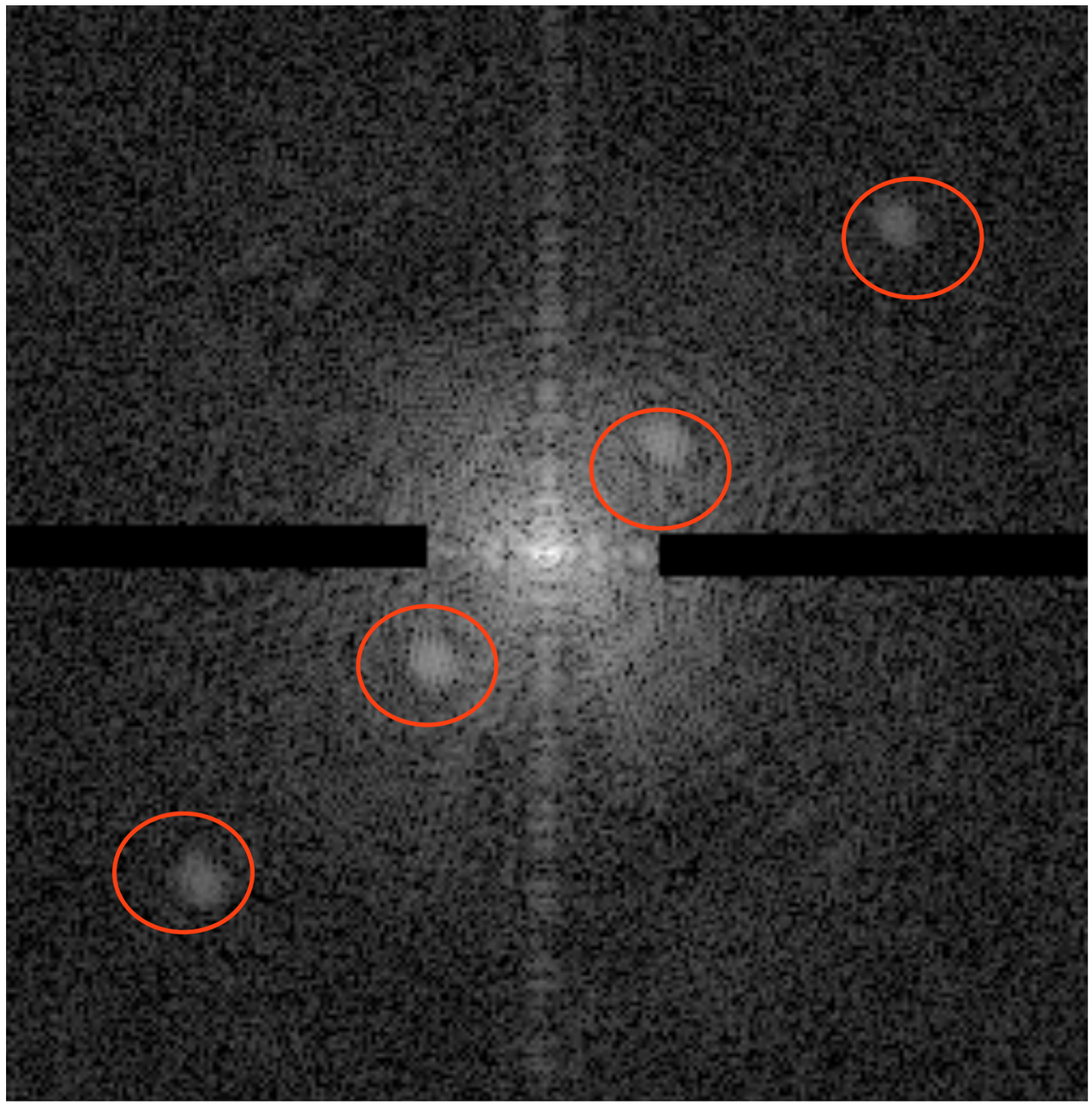
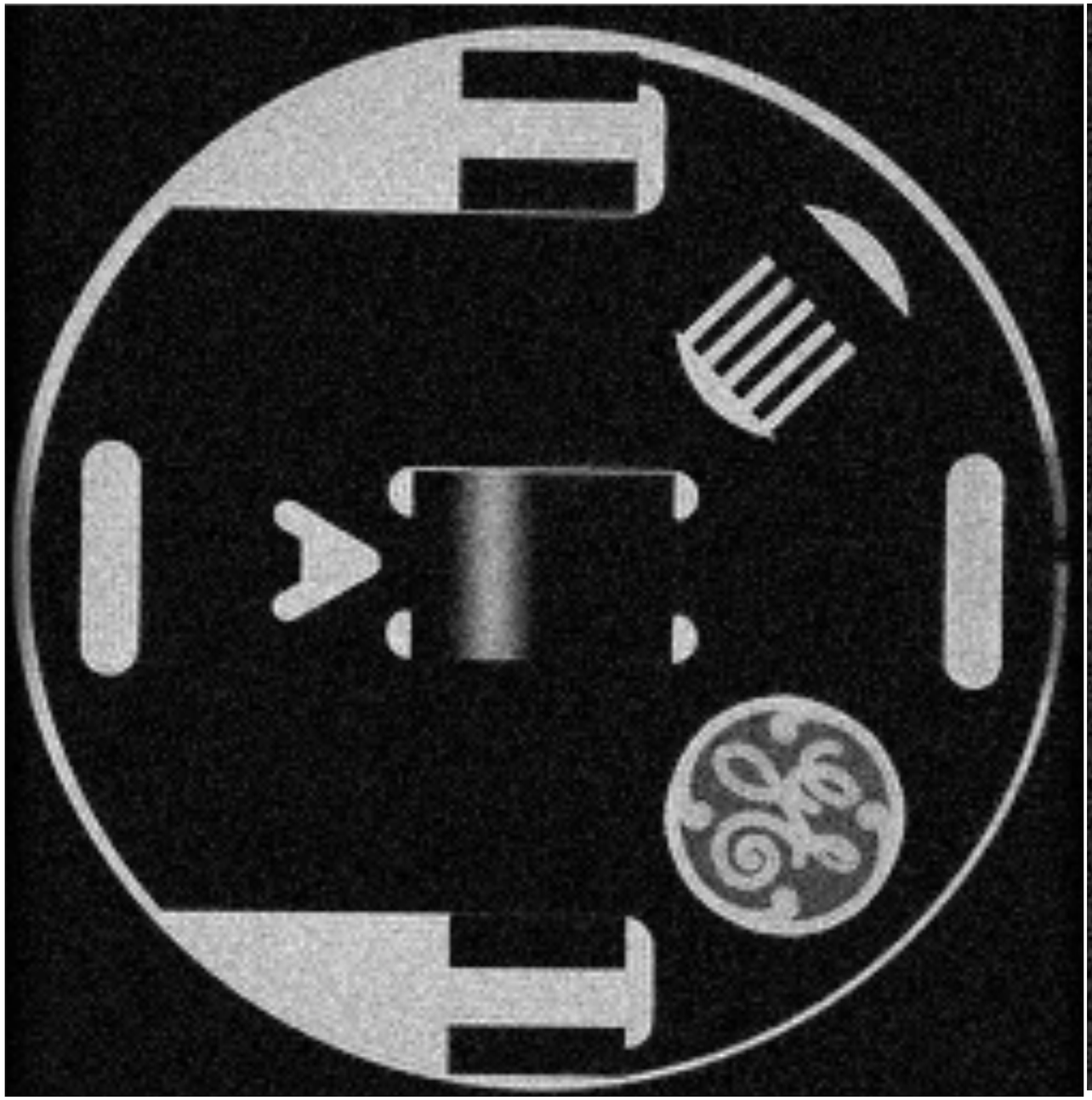
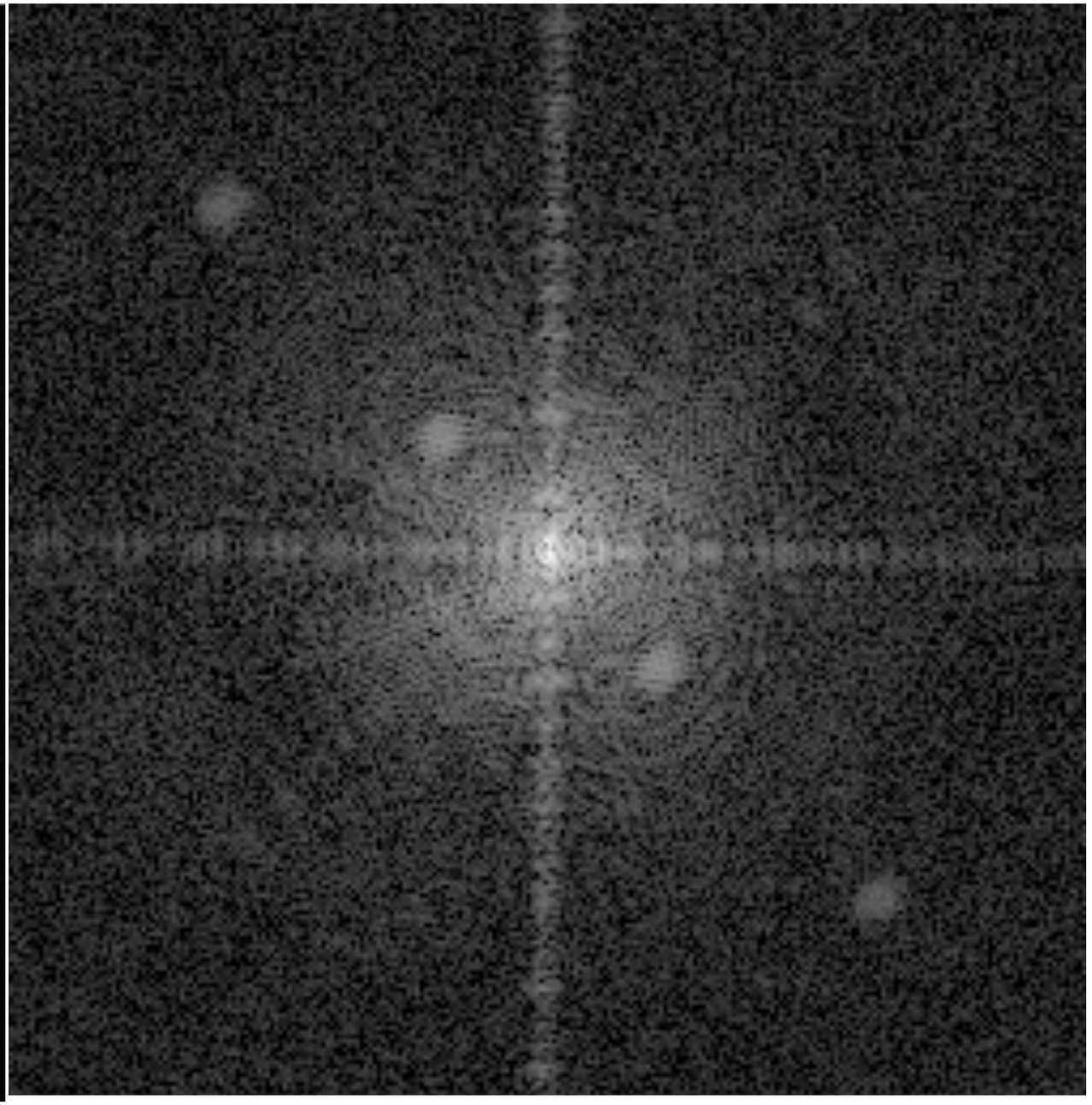
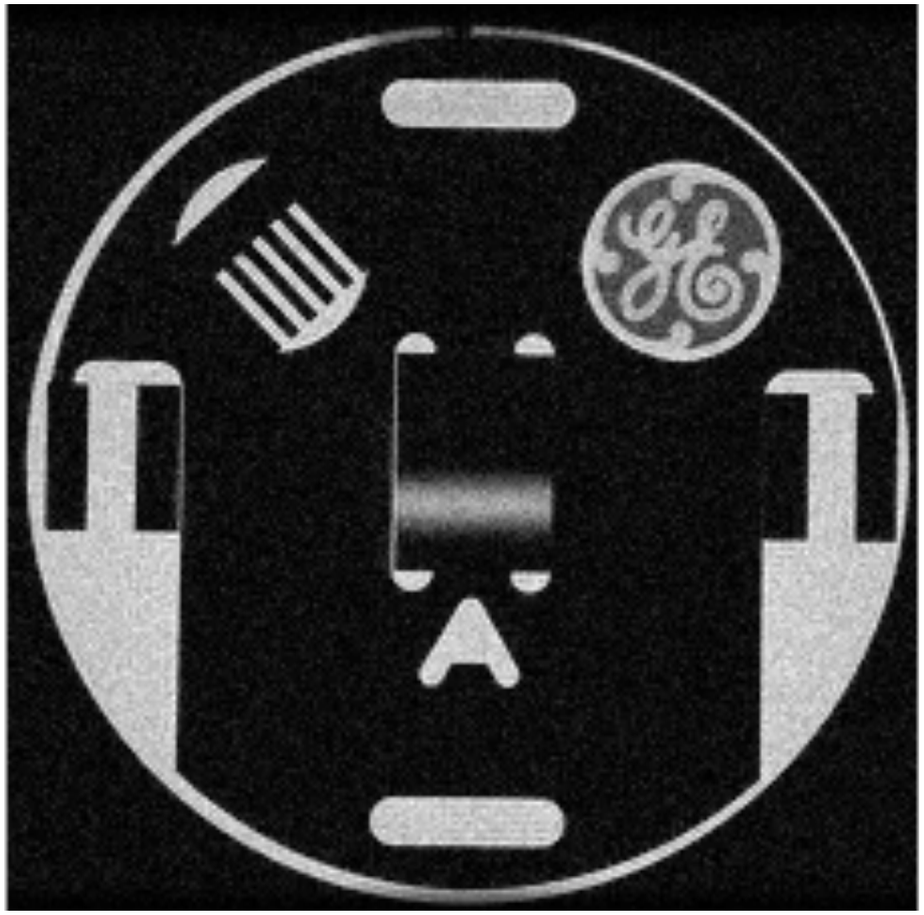
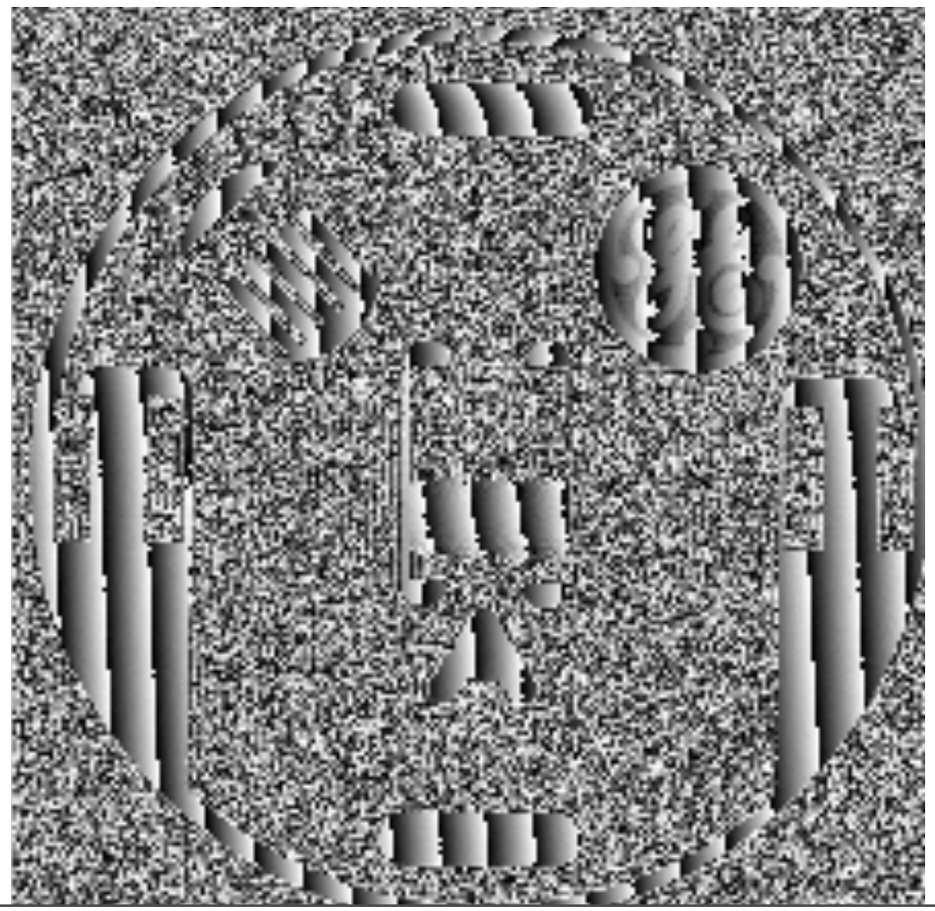
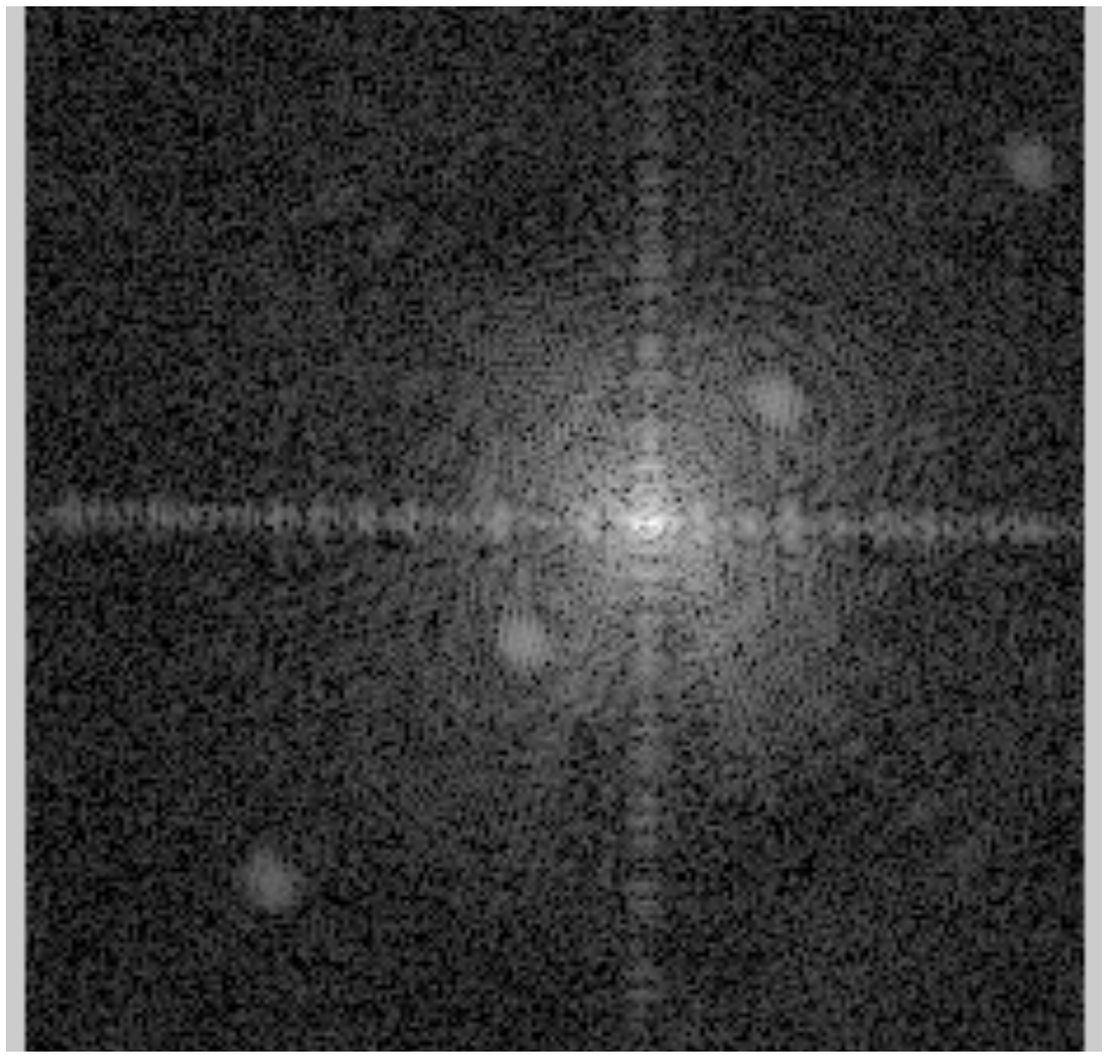




$$\frac{J_1(\pi r)}{2r}$$



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