Lab #3 Solutions

1. The consistency condition holds if and only if

$$\exists\, x \;:\; \|x - x_i\|_2 \le R_i, \;\; i = 1, \ldots, m.$$

We can formulate this as an optimization problem with constant, for example zero, objective:

$$\min_x \; 0 \;:\; \|x - x_i\|_2 \le R_i, \;\; i = 1, \ldots, m.$$

The above is an SOCP. On exit, a solver such as CVX will provide either a feasible point x_f, or determine (unambiguously) that there is no feasible point.

```
cvx_begin
    variable xfeas(2,1)
    minimize( 0 )
    subject to
        for i = 1:m,
R(i) >= norm(xfeas-X(:,i),2);
end
cvx_end
if ~isfinite(cvx_optval), xfeas = []; end
```

2. As seen in Fig. 1, with one measurement added, the data set is inconsistent.

We solve the problem

$$\min_{x,\delta} \; \|\delta\| \;:\; \delta \ge 0, \;\; \|x - x_i\|_2 \le R_i + \delta_i, \;\; i = 1, \ldots, 4,$$

with the norm $\|\cdot\|$ being the Euclidean norm and then the l_1-norm. We may ignore the sign constraint on δ if we do not want to assume that the errors are over-estimation errors only. The above problem can be expressed as an SOCP in both cases. The qualitative behavior of the solution depends on the choice of the norm.

The following snippet assumes that the integer $p \in \{1, 2\}$ exists in matlab's workspace.

```
cvx_begin
        variable delta(m+1,1)
        variable x(2,1)
        minimize( norm(delta,p) )
```

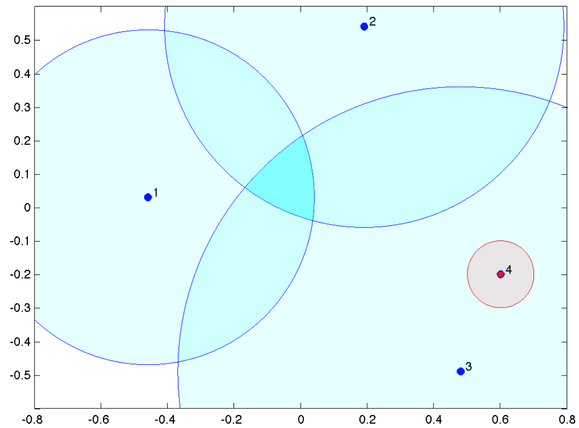


Figure 1: Inconsistent data set.

```
        subject to
            for i = 1:m+1,
                R(i) + delta(i)>= norm(x-X(:,i),2);
            end
            %delta >= 0; % sign constraints disabled
cvx_end
```

Choosing an l_1 norm will tend to make the smallest number of adjustment necessary. This would make sense if we believe that a few of our measurements are outliers, and due to faulty sensors.

4. We first focus on the problem of finding the largest radius of a sphere contained in the intersection. It is easy to check that a sphere of center x_0 and radius R_0 is contained in a sphere of center x_i and radius R_i if and only if the differences in the radiuses exceeds the distance between the centers:

$$R_i \geq R_0 + \|x_i - x_0\|_2.$$

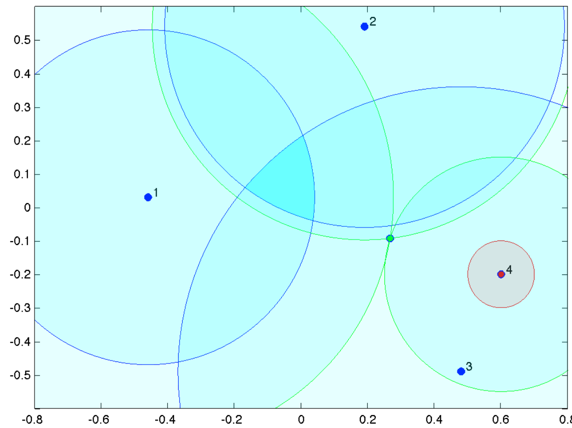
Our inner approximation problem then becomes the SOCP

$$\max_{x_0, R_0} R_0 \;:\; R_i \geq R_0 + \|x_i - x_0\|_2, \;\; i = 1, \ldots, m.$$

We note that the measurements are inconsistent if and only if at optimum, $R_0^* < 0$. This is the same as saying that there is no point x_0 which satisfies the constraints $\|x_i - x_0\|_2 \leq R_i$, $i = 1, \ldots, m$.

5. We simply ensure that the vertices of a box with size ρ are inside the intersection, and then maximize ρ. In 2D or 3D, this is easy, as there is a moderate number of vertices. The problem is written

$$\max_{\rho, x_0} \rho \;:\; \|x_0 + \rho v_k - x_i\|_2 \leq R_i, \;\; i = 1, \ldots, m, \;\; k = 1, \ldots, K.$$



3. **Figure 2:** If the norm chosen is the Euclidean one, we are minimizing the sum of the squares of the adjustments (increases) that are necessary to make our measurements consistent. This results in non-zero adjustments for all the measurements, and the new (unique) intersection point (in green) is far away from the initial intersection.

In the above, K is the number of vertices of the box ($K = 2$ in 2D, $K = 8$ in 3D), and v_k, $k = 1, \ldots, K$ are the vertices of the unit box, that is, the vectors with elements ± 1.

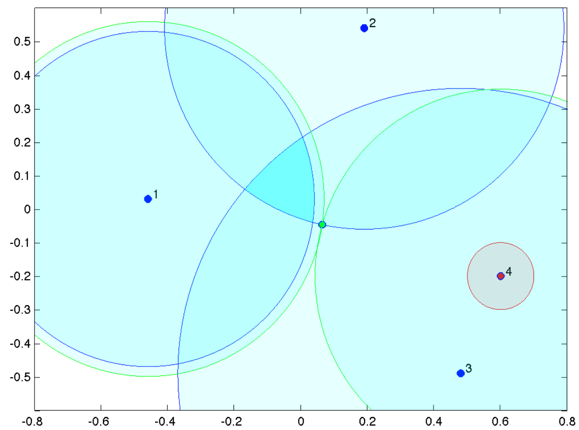


Figure 3: Identifying a faulty sensor that resulted in inconsistent measurements: here we have solved the minimum l_1-norm of adjustments necessary to make the measurements consistent. The approach identifies the offending measurement, since the optimal adjustment vector δ is almost zero except for its last component, which corresponds to the fourth sensor. We observe, however, that the indentification is not perfect, as the measurement of sensor 1 is also adjusted, albeit only slightly. This is due to the fact that the l_1-norm approach is a only a heuristic to solve cardinality minimization problems.

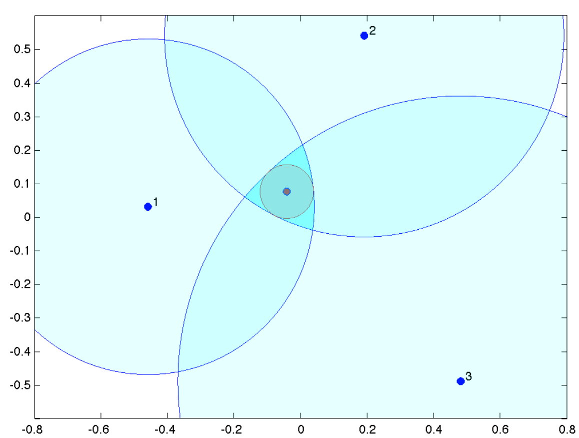


Figure 4: Inner spherical approximation to the intersection. This provides an estimated point (the center of the inner shpere), with an /optimistic/ estimate of the uncertainty around it.

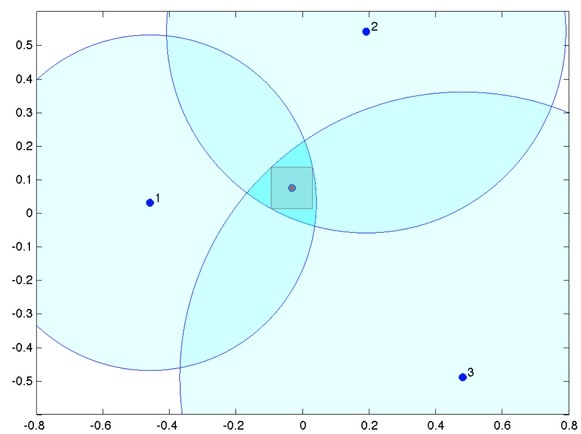


Figure 5: Inner box approximation to the intersection. This provides an estimated point (the center of the inner box), with an /optimistic/ estimate of the uncertainty around it. Here, the uncertainty is given as two intervals of confidence on each of the coordinates of the estimated point.