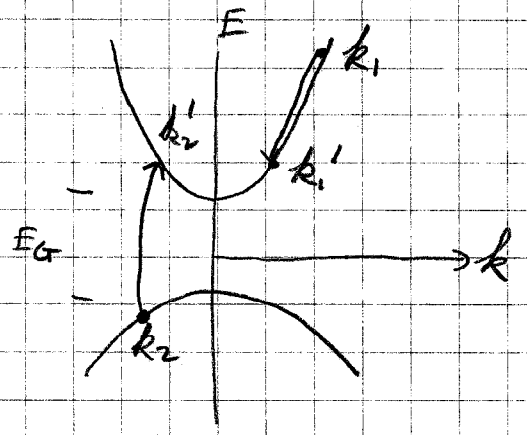
Problem 1

a)



initial electron momentum $\hbar k_1$
final electron momentum $\hbar k_1'$
generated hole momentum $\hbar k_2$
generated electron momentum $\hbar k_2'$

In the initial state, we only have an electron with momentum $\hbar k_1$. In the final state we have 2 electrons and a hole with momenta $\hbar k_1'$, $\hbar k_2'$, and $\hbar k_2$, respectively.

Conservation of momentum says:

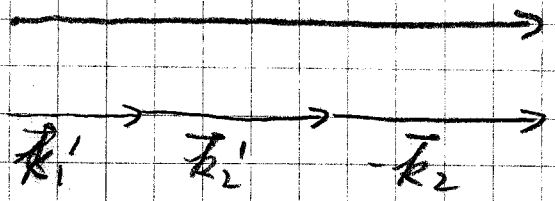
$$\hbar k_1 = \hbar k_1' + \hbar k_2' - \hbar k_2 \qquad (1)$$

↑ because this is a hole

If the 3 final particles are at rest, then k_1', k_2' and k_2 are all zero, which violates eqn. (1).

b) Clearly we must increase $\hbar k_1$ in order to get the final momenta to add up.

c) We need to find the minimum $\hbar k_1 + \hbar k_1' + \hbar k_2$ that satisfies energy and momentum conservation. Since k_1 is the largest vector, we will line up the other 3 vectors along k_1 as shown



so: $k_1 = k_1' + k_2' + k_2$

let $k_1' = a k_2$

$k_2' = b k_2$

then $k_1 = (a+b+1) k_2$

Conservation of energy states that

$$\frac{\hbar^2}{2m_c^*} k_1^2 = \frac{\hbar^2}{2m_c^*}\left(k_1'^2 + k_2'^2\right) + \frac{\hbar^2}{2m_v^*} k_2^2 + E_g$$

this can be re-written

$$k_1^2 = \mu k_2^2 (a^2 + b^2 + \mu) + k_g^2$$

where $\mu = \frac{m_c^*}{m_v^*}$ and $\frac{\hbar^2 k_g^2}{2m_c^*} = E_g$

also cons. of momentum gives

$$k_1^2 = (a+b+1)^2 k_2^2 = (a^2 + b^2 + 1 + 2ab + 2a + 2b) k_2^2$$

thus $(1 + 2ab + 2a + 2b - \mu) k_2^2 = k_g^2$ (2)

We need to find the minimum initial energy, $\frac{\hbar^2 k_1^2}{2m_c^*}$.

This is equivalent to minimizing: $(k_1'^2 + k_2'^2 + \mu k_2^2)$, or $(a^2 + b^2 + \mu) k_2^2$.

From eqn ②:

$$k_2^2 = \frac{k_g^2}{1+2ab+2a+2b-\mu}$$

So we need to minimize

$$\frac{(a^2+b^2+\mu)k_g^2}{1+2ab+2a+2b-\mu} \equiv Q$$

Q is symmetric in a and b so it must be that $a=b$ then we minimize

$$Q' = \frac{2a^2+\mu}{1+2a^2+4a-\mu} \quad \text{w.r.t. } a$$

$$\frac{\partial Q'}{\partial a} = \frac{4a(1+2a^2+4a-\mu) - (2a^2+\mu)(4a+4)}{(1+2a^2+4a-\mu)^2}$$

$$4a+8a^3+16a^2-4a\mu-8a^3-4a\mu-8a^2-4\mu = 0$$

$$8a^2+(4-8\mu)a-4\mu = 0$$

$$a = \frac{(8\mu-4) \pm \sqrt{(8\mu-4)^2+128\mu}}{16}$$

$$= \frac{8\mu-4 \pm \sqrt{64\mu^2+64\mu+16}}{16}$$

$$= \frac{8\mu-4 \pm (8\mu+4)}{16} = +\mu \quad \text{(since } a \text{ must be positive)}$$

So $k_2^2 = \dfrac{k_g^2}{1+2\mu^2+3\mu}$

$$k_1^2 = (2\mu^2+\mu)k_2^2 + k_g^2 = k_g^2\left[1+\frac{2\mu^2+\mu}{2\mu^2+3\mu+1}\right] = k_g^2\left[1+\frac{\mu(2\mu+1)}{(\mu+1)(2\mu+1)}\right]$$

$$k_1^2 = k_g^2\left[\frac{2\mu+1}{\mu+1}\right] \quad \text{thus} \quad \boxed{E_1 = \left[\frac{2\mu+1}{\mu+1}\right]E_G}$$

if $\mu=1$, $E_1 = \frac{3}{2}E_G$

Problem 2

From class notes:

$$\vec{J}_n = n e \mu \vec{E} + e D_n \vec{\nabla} n + e S_n \vec{\nabla} T_e$$

assuming uniform n, and neglecting electronic heat diffusion,

$$J_x = n e \mu E_x .$$

From notes, energy flux is

$$\vec{F}_w = \frac{-5}{2} \frac{k T_e}{e} \vec{J} - \kappa \vec{\nabla} T_e$$

$$= -\frac{5}{2} \frac{k T_e}{e} J$$

again, neglecting electronic heat diffusion

The energy balance equation is: (from class)

$$\frac{3}{2} n k \frac{T_e - T_0}{\tau_w} = \vec{J} \cdot \vec{E} - \vec{\nabla} \cdot \vec{F}_w$$

$$= n e \mu E_x^2 + \frac{5}{2} \frac{d}{dx} \left(\frac{k T_e}{e} J \right)$$

Since J must be constant $\left(\frac{d}{dx} J = 0 \right)$, we get

$$\frac{3}{2} n k \frac{T_e - T_0}{\tau_w} = n e \mu E_x^2 + \frac{5}{2} n k \mu E_x \frac{dT_e}{dx}$$

$$\frac{3}{5} \frac{(T_e - T_0)}{\mu E_x \tau_w} - \frac{dT_e}{dx} = \frac{2}{5} \frac{e}{k} E_x$$

now let $\mu = \mu_0 \frac{T_0}{T_e}$

$$\boxed{\frac{3}{5}\frac{T_e(T_e-T_0)}{\mu_0 T_0 E_x \tau_w} - \frac{dT_e}{dx} = \frac{2}{5}\frac{e}{k}E_x}$$

b). Try $T_e(x) = \frac{T_0}{2}[1 + A\tanh(Bx+c)]$

$$\frac{dT_e}{dx} = \frac{T_0 AB}{2}\operatorname{sech}^2(Bx+c)$$

then

$$\frac{3}{5}\frac{1}{\mu_0 T_0 E_x \tau_w}\frac{T_0}{2}(1+A\tanh(Bx+c))\left[\frac{T_0}{2}(A\tanh(Bx+c))-1\right]$$

$$-\frac{T_0 AB}{2}\operatorname{sech}^2(Bx+c) = \frac{2}{5}\frac{e}{k}E_x$$

$$\frac{3}{20}\frac{T_0}{\mu_0 E_x \tau_w}\left[A^2\tanh^2(Bx+c)-1\right] - \frac{T_0 AB}{2}\operatorname{sech}^2(Bx+c) = \frac{2}{5}\frac{e}{k}E_x$$

$$\frac{3}{20}\frac{A^2 T_0}{\mu_0 E_x \tau_w}\tanh^2(Bx+c) - \frac{T_0 AB}{2}\operatorname{sech}^2(Bx+c) = \frac{2}{5}\frac{e}{k}E_x + \frac{3T_0}{20\mu_0 E_x \tau_w}$$

Since $\tanh^2 y + \operatorname{sech}^2 y = 1$,
we can write:

$$A^2 = \frac{20\mu_0 E_x \tau_w}{3T_0}\left[\frac{2}{5}\frac{e}{k}E_x + \frac{3}{20}\frac{T_0}{\mu_0 E_x \tau_w}\right]$$

$$\boxed{A = \left[1 + \frac{8}{3}\frac{e}{kT_0}\mu_0 \tau_w E_x^2\right]^{1/2}}$$

also $\quad \frac{T_0 AB}{2} = \frac{-3}{20}\frac{A^2 T_0}{\mu_0 E_x \tau_w}$

$$B = -\frac{3}{10}\frac{A}{\mu_0 E_x \tau_w}$$

$$= \frac{-3}{10\mu_0 E_x \tau_w}\left(1 + \frac{8}{3}\frac{e}{kT_0}\mu_0 \tau_w E_x^2\right)^{1/2}$$

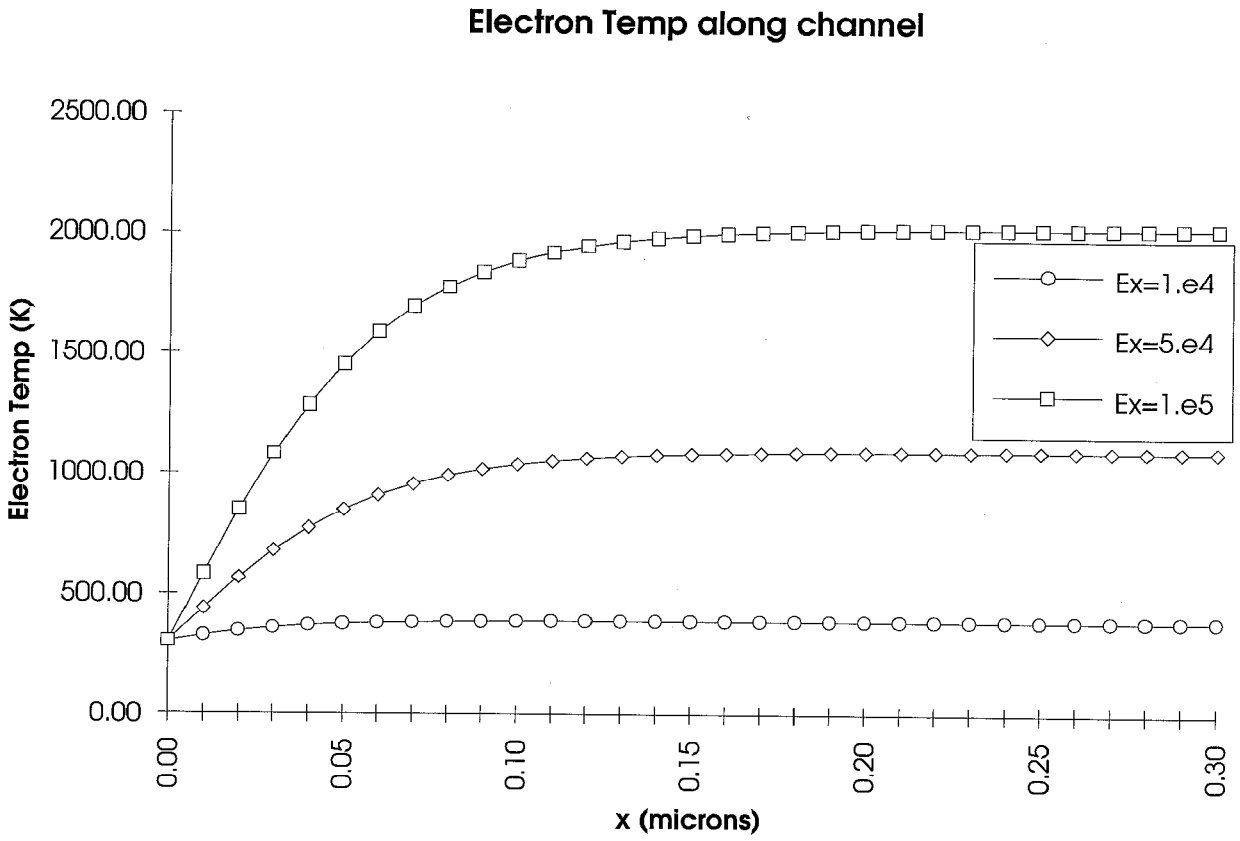
$$\boxed{B = -\frac{1}{E_x}\left[\left(\frac{3}{10\mu_0 \tau_w}\right)^2 + \frac{6}{25}\frac{e}{kT_0}\frac{E_x^2}{\mu_0 \tau_w}\right]^{1/2}}$$

from Boundary condition, $T_e(x=0) = T_0$

$$T_0 = \frac{T_0}{2}(1 + A\tanh c)$$

$$\frac{1}{2} = \frac{A}{2}\tanh c$$

$$\boxed{c = \tanh^{-1}\frac{1}{A}}$$


Electron Temp along channel
2500.00
2000.00
1500.00
1000.00
500.00
0.00
Electron Temp (K)
0.00
0.05
0.10
0.15
0.20
0.25
0.30
x (microns)
Ex=1.e4
Ex=5.e4
Ex=1.e5

d)

$$V_d = -\mu E_x = -\mu_0 \frac{T_0}{T_e} \quad \text{for electrons}$$

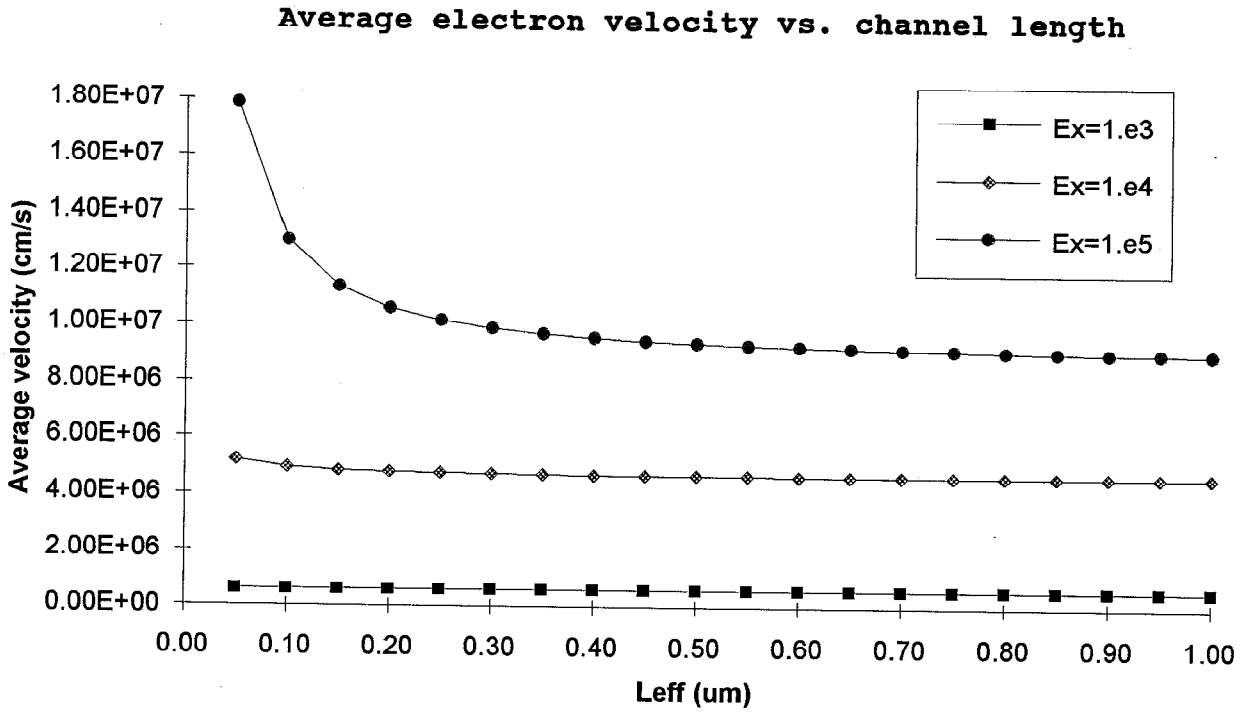
$$t_{transit} = \int_0^{L_{eff}} \frac{1}{V_d}\, dx$$

$$= \frac{-1}{\mu_0 T_0 E_{x0}} \int_0^{L_{eff}} T_e(x)\, dx = \frac{-1}{2\mu_0 E_x} \int_0^{L_{eff}} [1 + A\tanh(Bx+C)]\, dx$$

$$= \frac{-1}{2\mu_0 E_x} \left[L_{eff} + \frac{A}{B} \int_C^{BL_{eff}+C} \tanh y\, dy \right]$$

$$= \frac{-1}{2\mu_0 E_x} \left[L_{eff} + \frac{A}{B} \ln \frac{\cosh(BL_{eff}+C)}{\cosh C} \right]$$

$$V_{av} = \frac{-2\mu_0 E_x L_{eff}}{L_{eff} + \frac{A}{B} \ln \left[\frac{\cosh(BL_{eff}+C)}{\cosh C} \right]}$$


Average electron velocity vs. channel length
Average velocity (cm/s)
1.80E+07
1.60E+07
1.40E+07
1.20E+07
1.00E+07
8.00E+06
6.00E+06
4.00E+06
2.00E+06
0.00E+00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
Leff (um)
Ex=1.e3
Ex=1.e4
Ex=1.e5

Problem 3

Over this small range of energy, the variation in luminescence intensity simply reflects the electron energy distribution. Hence the electron temperature can be extracted from the slope of the exponential tail of the luminescence spectrum.
In the tail region $I = I_0 e^{-\hbar\nu/kT}$

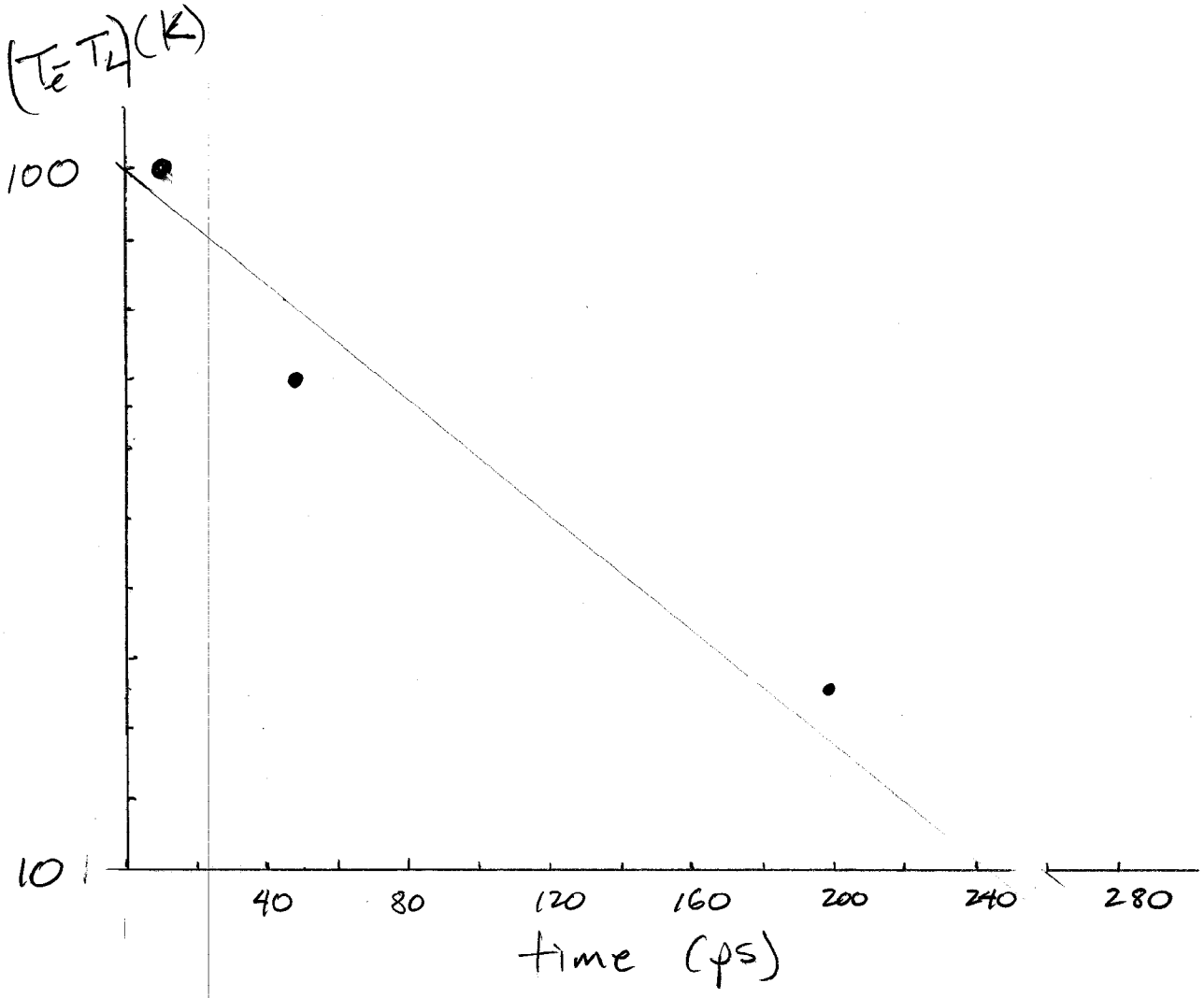
take $\log_{10} I = \log_{10} I_0 - \frac{\hbar\nu}{kT} \log_{10} e$

thus a drop in I by one decade occurs over an energy range $\Delta(\hbar\nu) = \frac{kT}{\log_{10} e}$

Read off the figure $\Delta(h\nu)_{10}$, the energy range to drop one decade in I. Then $kT = \log_e \Delta(h\nu)_{10}$

	$(\Delta\hbar\nu)_{10}$ (meV)	kT (meV)	T (K)
9 psec	22.2	9.64	111
50 psec	11.1	4.84	56
200 psec	5.5	2.4	28

The energy relaxation time is simply the exponential time constant for temperature cooling, back to $T_L = 10K$ (from figure)



From this plot, we see the decay is not quite exponential but a best fit gives a relaxation time of

$$\tau_E \approx 100 \text{ ps}$$

Problem 4

In high field, the balance equation tells us that heating is proportional to mobility. Therefore those valleys oriented such that m^* is light will heat preferentially over valleys oriented such that m^* is heavier. Thus, in general, except for special directions [(111) directions where all valleys have equal m^*], some valleys will be heated preferentially over others. This will lead to an imbalance in the intervalley scattering rates, and hence unequal populations in the equivalent valleys. Unequal populations then lead to current that is no longer parallel to the field.

Problem 5

a) Strongly dispersive means the electrons are mobile and can move freely with an effective mass given by the band curvature. A flat band corresponds to essentially infinite effective mass, which means that the electrons are not mobile, they are localized a The fact that the band is dispersive along Gamma-J but flat along J-K' means that the electrons are free to move along the chains on the surface, but they do not move in a direction perpendicular to the chains.
Each chain is a 1-D conductor and there is essentially no conductance for electrons moving across from chain to chain.

b) In 3D, we had

$$R(\nu)d\nu = \alpha(\nu)\frac{c}{n}\rho(\nu)d\nu$$

$R(\nu)$: emission rate/unit volume
$\alpha(\nu)$: absorption/unit length
$\rho(\nu)$: blackbody photons/unit volume

In this problem, we are given

$\beta(\nu)$: the (%) absorption per 2D layer, (a dimensionless quantity).

We can understand this by going to the concept of the absorption <u>cross section</u> per "atom", $\sigma(\nu)$. $\sigma(\nu)$ has dimensions of area. In terms of $\sigma(\nu)$, the 3D absorption coefficient is $\alpha(\nu) = \sigma(\nu) N_{3D}$, where N_{3D} is the number of "atoms" per unit volume. $\alpha(\nu)$ has dimensions of 1/length. For a single, 2D layer of atoms, the absorption $\beta(\nu) = \sigma(\nu) N_{2D}$, with N_{2D} the number of "atoms" per unit area. $\beta(\nu)$ is dimensionless. The 2D layer simply absorbs some fraction of the incident beam.

We therefore write

$$R(\nu) = \beta(\nu)\frac{c}{n}\rho(\nu)d\nu,$$

where now, $R(\nu)$ is the emission rate/unit <u>area</u>.

So $R(\nu)d\nu = \dfrac{\beta(\nu)\,8\pi\nu^2 n^2}{c^2\left(e^{h\nu/kT}-1\right)}\,d\nu$

We are given $\beta(E)dE$ $(E=h\nu)$, so we rewrite

$$R(E)dE = \frac{\beta(E)\,8\pi E^2 n^2}{h^3c^2\left(e^{E/kT}-1\right)}\,dE$$

$$R_0 = \int_0^\infty R(E)dE = \frac{8\pi n^2}{h^3c^2}\int_0^\infty \frac{\beta(E)E^2\,dE}{e^{E/kT}-1}$$

$\beta(E)$ is approximately a Gaussian

$$\beta(E) = \beta_0 e^{-a^2(E-E_0)^2} \qquad E_0 = 0.47\ eV\ ;\ \beta_0 = 0.02$$

Since the FWHM is 0.1eV, then $a = \dfrac{\sqrt{\ln 2}}{0.05\,eV} = 16.65\,eV$

$$R_0 = \frac{8\pi n^2}{h^3c^2}\beta_0\int_0^\infty \frac{e^{-a^2(E-E_0)^2}E^2dE}{e^{E/kT}-1}$$

I evaluated the integral numerically:

$$\int_0^\infty \frac{e^{-a^2(E-E_0)^2}E^2dE}{e^{E/kT}-1} = 5\times10^{-10}\,(eV)^3$$

For Si, $n^2 \cong 12$

So $$\boxed{R_0 = \frac{8\pi(12)(.02)\left(5\times10^{-10}eV^3\right)}{\left(4.1\times10^{-15}eV\text{-}s\right)^3\left(3\times10^{10}cm/s\right)^2} = 4.9\times10^{13}\ cm^{-2}s^{-1}}$$

c) We must calculate the surface band occupation densities using the 1D density of states.

$$g_{1D}(E) = \frac{(2m^*)^{1/2}}{\pi\hbar}(E-E_c)^{-1/2}$$

surface π^* "conduction band" density of states per unit length.

E_c = energy of bottom of π^* band.

The occupation probability is given by Fermi-Dirac statistics:

$$f(E) = \frac{1}{1+\exp[(E-E_F)/kT]}$$

Then the π^* density/unit length per chain is

$$n_c = \frac{(2m^*)^{1/2}}{\pi\hbar}\int_0^\infty \frac{(E-E_c)^{-1/2}\,dE}{1+\exp[(E-E_f)/kT]}$$

Using reduced variables:

$$\eta = \frac{(E-E_c)}{kT} \quad \eta_c = \frac{E_c}{kT} \quad \mu = \frac{E_F}{kT}$$

$$n_c = \frac{(2m^*kT)^{1/2}}{\pi\hbar}\int_0^\infty \frac{\eta^{-1/2}\,d\eta}{1+\exp(\eta-\mu+\eta_c)}$$

With the definition of the F-D integral:

$$F_j(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{y^j\,dy}{1+\exp(y-x)},$$

we have:

$$n_c = \left(\frac{m^*kT}{2\pi\hbar^2}\right)^{1/2} F_{-1/2}(\mu-\eta_c)$$

In the non-degenerate limit, the Fermi-Dirac integral is:

$$F_j(x) \cong \frac{2}{\sqrt{\pi}} \int_0^\infty y^j e^{x-y}\, dy$$

$$= \frac{2}{\sqrt{\pi}} e^x \Gamma(j+1) \qquad \text{from integral tables}$$

also $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ Math tables

so $F_{-1/2}(\mu - \eta_c) = 2e^{(\mu-\eta_c)}$.

Then $n_e = \left(\frac{2m^* kT}{\pi\hbar^2}\right)^{1/2} e^{\mu-\eta_c}$.

For the holes in the π band, we get

$$p_e = \left(\frac{2m^* kT}{\pi\hbar^2}\right)^{1/2} e^{\eta_v - \mu}$$

with $\eta_v = \frac{E_v}{kT}$; E_v = energy of top of π band.

Then, in terms of these "linear" densities

$$n_e p_e = \frac{2m^* kT}{\pi\hbar^2} e^{-(\eta_c - \eta_v)} = \frac{2m^* kT}{\pi\hbar^2} e^{-E_g/kT}$$

To get the areal densities, we divide n_e and n_p by d_c, the interchain spacing ($d_c = 6.65\,A^\circ$).

Then $\boxed{n_i^2 = \frac{2m^* kT}{\pi\hbar^2 d_c^2} e^{-E_g/kT}}$

Using $m^* = 0.16\, m_e = 1.45\times10^{-31}$ kg

$E_g = 0.45$ eV $= 7.2\times10^{-20}$ J

$kT = 26$ meV $= 4.1\times10^{-21}$ J

$d_c = 6.65\,A^\circ = 6.65\times10^{-10}$ m

$$n_i^2 = \frac{2(1.45\times10^{-31}\,\text{kg})(4.1\times10^{-21}\,\text{J})}{\pi(1.05\times10^{-34}\,\text{J-s})^2(6.65\times10^{-10}\,\text{m})^2}\, e^{-0.45/.026}$$

$$= 2.4\times10^{27}\ \frac{\text{kg J}}{\text{J}^2\text{s}^2\,\text{m}^2} \qquad \text{J} = \frac{\text{kg m}^2}{\text{s}^2}$$

$$n_i^2 = 2.4\times10^{27}\ \text{m}^{-4}\ ;\ n_i = 4.9\times10^{13}\ \text{m}^{-2}$$

$$n_i^2 = 2.4\times10^{19}\ \text{cm}^{-4}\ ;\ n_i = 4.9\times10^{9}\ \text{cm}^{-2}$$

d)

$$R = \frac{R_o}{n_i^2}\, np$$

With the Fermi level pinned at E_c, the surface is clearly n-type, with $n_o \gg p_o$

For excess surface holes $\Delta p \ll n_o$

$$R = \frac{R_o}{n_o p_o}(n_o+\Delta n)(p_o+\Delta p) \cong \frac{R_o}{n_o p_o}\, n_o \Delta p = \frac{R_o}{p_o}\Delta p$$

$$\frac{d\Delta p}{dt} = -\frac{R_o}{p_o}\Delta p \equiv -\frac{\Delta p}{\tau} \qquad \tau = \frac{p_o}{R_o}$$

$$p_o = \left(\frac{2m^*kT}{\pi\hbar^2}\right)^{1/2} e^{-E_g/kT} = 8.6\times10^{5}\ \text{cm}^{-2}$$

$$\therefore\ \tau = \frac{p_o}{R_o} = \frac{8.6\times10^{5}\ \text{cm}^{-2}}{4.9\times10^{13}\ \text{cm}^{-2}\text{s}^{-1}} = 1.76\times10^{-8}\,\text{s} = 17.6\,\text{ns}$$