EE236
Problem set 1
Fall 2004

1) Show that if an Operator is Hermitian, then a matrix which represents it is Hermitian, and vis versa

The definition of a Hermitian operator is
$$\langle \psi | A^{+} | \xi \rangle = \langle A\psi | \xi \rangle = \langle \psi | A | \xi \rangle$$
for arbitrary $|\psi\rangle$, $|\xi\rangle$

For any matrix representation must be a representation over a complete set of states $|\phi_n\rangle$
so we have the identity
$$I = \sum_m |\phi_m\rangle\langle\phi_m|$$

the Matrix which represents A in this basis is
$$a_{mn} = \langle \phi_m | \hat{A} | \phi_n \rangle$$

$$a_{nm} = \langle \phi_n | \hat{A} | \phi_m \rangle$$
$$a_{nm} = \langle A^{+}\phi_n | \phi_m \rangle = \langle A\phi_n | \phi_m \rangle$$
$$= \langle \phi_m | A\phi_n \rangle^{*} = a_{nm}^{*}$$

Now we show the reverse

$$a_{nm} = a_{mn}^*$$

we have $\langle \phi_m | A | \phi_n \rangle = \langle \phi_n | A | \phi_m \rangle^*$

starting with

$$\langle \psi | A | \xi \rangle$$

$$= \sum_m \sum_n \langle \psi | \phi_m \rangle \langle \phi_m | A | \phi_n \rangle \langle \phi_n | \xi \rangle$$

$$= \sum_m \sum_n \langle \psi | \phi_m \rangle \langle \phi_n | A | \phi_m \rangle^* \langle \phi_n | \xi \rangle$$

$$= \sum_m \sum_n \langle \psi | \phi_m \rangle \langle A \phi_m | \phi_n \rangle \langle \phi_n | \xi \rangle$$

$$= \sum_m \sum_n \langle \psi | \phi_m \rangle \langle \phi_m | A^\dagger | \phi_n \rangle \langle \phi_n | \xi \rangle$$

$$= \langle \psi | A^\dagger | \xi \rangle$$

for arbitrary $|\psi\rangle$ & $|\xi\rangle$

2) $[\hat{A}\hat{B}]_{mn}$

$$= \langle \phi_m | \hat{A}\hat{B} | \phi_n \rangle$$

Using the Identity $\hat{I} = \sum_K |\phi_K\rangle\langle\phi_K|$

$$[\hat{A}\hat{B}]_{mn} = \langle \phi_m | \hat{A}\hat{I}\hat{B} | \phi_m \rangle$$

$$[\hat{A}\hat{B}]_{mn} = \sum_K \langle \phi_m | \hat{A} | \phi_K \rangle \langle \phi_K | \hat{B} | \phi_m \rangle$$

$$[AB]_{mn} = \sum_K [\hat{A}]_{mK} [\hat{B}]_{Kn}$$

which is the definition of matrix multiplication

3) We must show $\langle \psi | \hat{A}\hat{B} | \phi \rangle$
$= \langle \psi | \hat{B}\hat{A} | \phi \rangle$

for arbitrary $|\psi\rangle$ & $|\phi\rangle$

Use the Identity Operator three times

$$\langle \psi | \hat{I} \hat{A} \hat{I} \hat{B} \hat{I} | \phi \rangle$$
$$= \sum_j \sum_K \sum_L \langle \psi | \phi_j \rangle \langle \phi_j | \hat{A} | \phi_K \rangle \langle \phi_K | \hat{B} | \phi_L \rangle \langle \phi_L | \phi \rangle$$

Since the Matrices commute,

$$\sum_K [\hat{A}]_{jK} [\hat{B}]_{KL} = [\hat{B}]_{jK} [\hat{A}]_{KL}$$

so from the (*) above

$$\langle \psi | \hat{I} \hat{A} \hat{I} \hat{B} \hat{I} | \phi \rangle = \sum_j \sum_K \sum_L \langle \psi | \phi_j \rangle \langle \phi_j | \hat{B} | \phi_K \rangle \langle \phi_K | \hat{A} | \phi_L \rangle \langle \phi_L | \phi \rangle$$

$$= \langle \psi | \hat{I} \hat{B} \hat{I} \hat{A} \hat{I} | \phi \rangle$$

$$\langle \psi | \hat{A} \hat{B} | \phi \rangle = \langle \psi | \hat{B} \hat{A} | \phi \rangle$$

for arbitrary $|\psi\rangle$ & $|\phi\rangle$

so $\hat{A}\hat{B} = \hat{B}\hat{A}$

if we start with $\hat{A}\hat{B} = \hat{B}\hat{A}$

we have $\langle \phi | \hat{A}\hat{B} | \psi \rangle = \langle \phi | \hat{B}\hat{A} | \psi \rangle$

for arbitrary $|\phi\rangle$ & $|\psi\rangle$ so in particular

$$\langle \phi_n | \hat{A} \hat{B} | \phi_m \rangle = \langle \phi_n | \hat{B} \hat{A} | \phi_m \rangle$$

inserting the identity $\langle \phi_n | \hat{A} \hat{I} \hat{B} | \phi_m \rangle$

$= \langle \phi_n | \hat{B} \hat{I} \hat{A} | \phi_m \rangle$

so $$\sum_K \langle \phi_n | \hat{A} | \phi_K \rangle \langle \phi_K | \hat{B} | \phi_m \rangle = \sum_K \langle \phi_n | \hat{B} | \phi_K \rangle \langle \phi_K | \hat{A} | \phi_m \rangle$$

so $$\sum_K [\hat{A}]_{nK} [B]_{Km} = \sum_K [\hat{B}]_{nK} [\hat{A}]_{Km}$$

and the matrices commute

4) The easiest finite basis is the eigenstates of the Hamiltonian. To make the math easier, I will use the box $0 \to a$ rather than the symmetric one done in class

$$\hat{H}\psi = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right] \psi(x) = i\hbar \frac{\partial \psi}{\partial t}$$

to which the solutions are

$$\psi = e^{-j\omega t} \sin Kx$$

$$Ka = n\pi \qquad K = \frac{n\pi}{a}$$

$$E = \hbar\omega = \frac{\hbar^2 K^2}{2m}$$

and the normalized eigenstates of the Hamiltonian are

$$|\phi_n\rangle \Rightarrow \phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right) \qquad n = 1, 2 \ldots$$

to find the matrix representation of the operator X, we use

$$[\hat{X}]_{mn} = \langle \phi_m | \hat{x} | \phi_n \rangle$$

$$[\hat{X}]_{mn} = \int_0^a \frac{2}{a} \sin\left(\frac{m\pi}{a} x\right) x \sin\left(\frac{n\pi}{a} x\right) dx$$

using the identity $\sin\alpha + \sin\beta = \frac{1}{2}\cos(\alpha - \beta) - \frac{1}{2}\cos(\alpha + \beta)$

$$[\hat{X}]_{mn} = \int_0^a \frac{1}{a} x \left[\cos\left(\frac{\pi}{a} x (m-n)\right) - \cos\left(\frac{\pi}{a} x (m+n)\right)\right] dx$$

$$\int x \cos(cx)\, dx = \frac{1}{c^2} \cos ax + \frac{x}{c} \sin(cx)$$

$$[\hat{X}]_{mn} = \frac{1}{a} \frac{a^2}{\pi^2 (m-n)} \cos\left(\frac{\pi}{a} x (m-n)\right)\Big|_0^a + \frac{x}{a} \frac{a^2}{\pi^2 (m-n)} \sin\left(\frac{\pi}{a} x (m-n)\right)\Big|_0^a$$

$$[\hat{X}_{mn}] = \frac{1}{a}\left(\frac{\pi}{a}(m-n)\right)^{-2} \cos\left(\frac{\pi}{a}x(m-n)\right)\Big|_0^a$$
$$+ \frac{x}{a}\left(\frac{\pi}{a}(m-n)\right)^{-1} \sin\left(\frac{\pi}{a}x(m-n)\right)\Big|_0^a$$
$$- \frac{1}{a}\left(\frac{\pi}{a}(m+n)\right)^{-2} \cos\left(\frac{\pi}{a}x(m+n)\right)\Big|_0^a$$
$$- \frac{x}{a}\left(\frac{\pi}{a}(m+n)\right)^{-1} \sin\left(\frac{\pi}{a}x(m+n)\right)\Big|_0^a$$

all of the sin terms are zero, so

$$[\hat{X}_{mn}] = \frac{a}{\pi^2}(m-n)^{-2}(\cos(\pi(m-n)-1)$$
$$- \frac{a}{\pi^2}(m+n)^{-2}(\cos\pi(m+n)-1)$$

if $m-n$ is even, so is $m+n$
and the cosine terms are 1
for $m-n$ odd, the cosine terms are -1

$$[\hat{X}_{mn}] = +\frac{a2}{\pi^2}\left((m+n)^{-2} - (m-n)^{-2}\right)$$

for $m-n$ odd $\quad$ $m-n$ even $\Rightarrow 0$
(for $m=n$) $\Rightarrow 0$ also

5) Using the expressions from problem 1, we have

$|\phi_1\rangle \Rightarrow \sqrt{\frac{2}{a}}\sin(\pi\frac{x}{a})$ (ground state)

$|\phi_2\rangle \Rightarrow \sqrt{\frac{2}{a}}\sin(2\pi\frac{x}{a})$

$$\psi(x,t) = \sqrt{\frac{2}{a}}\, e^{-i\omega_1 t}\sin(\pi\frac{x}{a}) + \sqrt{\frac{2}{a}}\, e^{-i\omega_2 t}\sin(2\pi\frac{x}{a})$$

where $E = \frac{\hbar^2 k^2}{2m}$ $\hbar\omega = E$

$$\omega_0 = \frac{1}{\hbar}\frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2$$

$$\omega_1 = \frac{1}{\hbar}\frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2$$

the probability distribution is given by

$P(x') = |\langle x'|\psi(t)\rangle|^2$ but $|x'\rangle$ is just a delta function

$$P(x') = \frac{2}{a}\left|\left(e^{-i\omega_1 t}\sin\pi\frac{x}{a}\right) + e^{-i\omega_2 t}\sin\left(2\pi\frac{x}{a}\right)\right|^2$$

6) after the measurement of an eigenvalue x', the state will be the eigenstate corresponding to that eigenvalue

$$|\psi\rangle \Rightarrow \delta(x-x')$$

to find the time evolution after this measurement, we would project this state over the eigenvectors of the Hamiltonian

$$|\psi(t=t_0)\rangle = \sum_{n=1}^{\infty} \langle \phi_n | \psi(t=t_0)\rangle \, |\phi_n\rangle$$

and then evolve them with time

$$|\psi(t)\rangle = \sum_{n=1}^{\infty} e^{-\omega_n(t-t_0)} \langle \phi_n | \psi(t=t_0)\rangle |\phi_n\rangle$$

$$\langle \phi_n | \psi(t=t_0)\rangle = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \delta(x-x')\, dx$$

$$= \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x'\right)$$