EE 236
problem set 4
solutions

1) Yariv 2.11

given $[q, p] = i\hbar$

$$qp - pq = i\hbar$$

$$qp = i\hbar + pq$$

$$pq = qp - i\hbar$$

we know $[q, p] = i\hbar$.

so

$$[q, p^1] = i\hbar \, 1 \, p^{(1-1)}$$

so $[q, p^l] = i\hbar \, l \, p^{l-1}$

for the particular case of $l=1$

$$[q, p^{l+1}] = q p^{l+1} - p^{l+1} q$$

$$= q \, p^l p^1 - p^l \, p \, q$$

assuming we know

$$[q, p^l] = i\hbar l p^{l-1}$$

for some particular l, as it is for 1 above:

$$q p^l - p^l q = i\hbar l p^{l-1}$$

$$[q, p^{l+1}] = (p^l q + i\hbar l p^{l-1}) p - p^l p q$$

$$= p^l (i\hbar + p q) + i\hbar l p^l - p^l p q$$

$$= p^l (i\hbar) + i\hbar l p^l$$

$$= i\hbar (l+1) p^l$$

so if the above expression is true for any integer, it is true for the successive integer, and since it is true for $l = 1$, it is true for all integers by induction.

The proof of

$$[p, q^l] = -i\hbar l q^{l-1}$$

is the same as the above with the formal substitutions $p \Leftrightarrow q$ $\hbar \Leftrightarrow -\hbar$

2) 2.12

assuming $F(\vec{p}, \vec{q})$ has a convergent taylor series about q_i, we write

$$\vec{F}(\vec{p}, \vec{q}) = \sum_n f(p_j, p_k, \vec{q})\, p_i^n$$

we then have

$$[q_i, F(\vec{p}, \vec{q})] = \sum_n f(p_j, p_k \vec{q})\, [q_i, p_i^n]$$

and from problem 2.11

$$[q_i, F(\vec{p}, \vec{q})] = \sum_n f(p_j, p_k \vec{q}) \cdot i\hbar n p_i^{n-1}$$

$$= \sum_n f(p_j, p_k, \vec{q})\, i\hbar \frac{\partial}{\partial p} p_i^n$$

$$= \frac{\partial}{\partial p_i} \sum_n f(p_j, p_k, \vec{q})\, i\hbar p_i^n$$

$$= i\hbar \frac{\partial}{\partial p_i} \frac{\partial F}{\partial p_i}$$

Once again $[p, F(\vec{q}, \vec{p})]$
$$= -i\hbar \frac{\partial F}{\partial q}.$$

is obtained by the formal substitutions

$$q_i \Leftrightarrow p_i$$
$$\hbar \Leftrightarrow -\hbar$$

3) Any solution to the H.O. can be expressed as an expansion over the eigenfunctions of the Hamiltonian

$$|\psi(t=0)\rangle = \sum_n C_n |\phi_n\rangle$$
$$|\psi(t)\rangle = \sum_n C_n |\phi_n\rangle e^{i\omega t}$$

where $\omega = \frac{E}{\hbar} = \frac{(n+1/2)\hbar\omega_0}{\hbar}$
$$= (n+1/2)\omega_0$$
$$|\psi(t)\rangle = \sum_n C_n |\phi_n\rangle e^{i(n+1/2)\omega_0 t}$$

if $\omega_0 \tau = 2\pi$

$$|\psi(t+\tau)\rangle = \sum_n C_n |\phi_n\rangle e^{i(n+1/2)(\omega_0 t + \omega_0 \tau)}$$

$$|\psi(t+\tau)\rangle = \sum_n C_n |\phi_n\rangle e^{i(n+1/2)\omega_0 t} e^{i(n+1/2)\omega_0 \tau}$$

$$= e^{i\frac{\omega_0 \tau}{2}} \sum_n C_n |\phi_n\rangle e^{i(n+1/2)\omega_0 t}$$

$$= e^{i\frac{\omega_0 \tau}{2}} |\psi(t)\rangle$$

if we find the expectation value of any operator $\hat{A}$

$$\langle \hat{A}(t+\tau)\rangle = \langle \psi(t+\tau)|\hat{A}|\psi(t+\tau)\rangle$$

$$= \langle \psi(t)| e^{-i\frac{\omega_0 \tau}{2}} \hat{A} e^{i\frac{\omega_0 \tau}{2}} |\psi(t)\rangle$$

$$= \langle \psi(t)|\hat{A}|\psi(t)\rangle$$

$$= \langle \hat{A}(t)\rangle$$

$\Rightarrow$ periodic with a period τ

It does not have to be sinusoidal, as seen in the previous problem, an observable can be any function of another observable operator

counterexample; $\hat{x}^2$

4) the position and the momentum are conjugal operators, with

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} K \hat{x}^2$$

and the raising and lowering operators

$$\hat{a}^{+} = \sqrt{\frac{C}{2\hbar\omega}} \left[\hat{V} - i\sqrt{\frac{L}{C}} \hat{I} \right]$$

$$\hat{a} = \sqrt{\frac{C}{2\hbar\omega}} \left[V + i\sqrt{\frac{L}{C}} \hat{I} \right]$$

so the wavefunctions in the $\hat{I}$ or $\hat{p}$ representations are the same as those in the $\hat{x}$ or $\hat{V}$ representations, when scaled in the momentum representation

$$\hat{p}_x \to p_x$$

$$\hat{x} \to i\hbar \frac{\partial}{\partial p_x}$$

the Hamitonian becomes

$$\hat{H} = \frac{p_x^2}{2m} + \frac{1}{2} K(i\hbar)^2 \frac{\partial^2}{\partial p_x^2}$$

and so we have

$$\frac{p_x^2}{2m} U_n(p_x) + \frac{1}{2} K(i\hbar)^2 \frac{\partial^2}{\partial p_x^2} U_n(p_x)$$

$$= E_n U_n(p_x)$$

$$-\frac{1}{2} K\hbar^2 \frac{\partial^2}{\partial p_x^2} U_n(p_x) + \frac{1}{2m} p_x^2 U(p_x) = EU$$