EE 236
Problem set 6
Solution

1) The approximate mode is a standing wave, so we can use Yariv 5.6-14

$$E_l(r,t) = -i\hat{x}\left(\frac{\hbar\omega_l}{V\varepsilon}\right)^{1/2}[a_l^+ - a]\sin k_l z$$

and the state of this mode is $|n\rangle$

So the RMS $\vec{E}$ field is

$$\sqrt{\langle |E|^2 \rangle}$$

$$= \left(\frac{\hbar\omega_l}{V\varepsilon}\right)^{1/2}\left(\langle n|(a_l^+ - a)^2|n\rangle\right)^{1/2} |\sin(k_l z)|$$

$$= \left(\frac{\hbar\omega_l}{V\varepsilon}\right)^{1/2}\left(\langle n|\cancel{\langle} a_l^+ a_l + a_l a_l^+|n\rangle\right)^{1/2} |\sin(k_l z)|$$

$$= \left(\frac{\hbar\omega_l}{V\varepsilon}\right)^{1/2}(2n+1)^{1/2}|\sin(k_l z)|$$

$\Rightarrow$ the peak occurs when $\sin(k_l z) = 1$

2) for a single lens followed by a distance d, we have

$$\begin{pmatrix} r_{out} \\ r'_{out} \end{pmatrix} = \begin{vmatrix} 1 & d \\ -1/f & 1-d/f \end{vmatrix} \begin{pmatrix} r_{in} \\ r'_{in} \end{pmatrix}$$

from 6.1-10 we get

$$r_{s+2} - 2b\, r_{s+1} + r_s = 0$$

where $b = \frac{1}{2}(A+D) = \frac{1}{2}(2 - d/f)$

$$b = \left(1 - \frac{d}{2f}\right)$$

substituting $r_s = \rho e^{is\theta}$

$$e^{2i\theta} - 2be^{i\theta} + 1 = 0$$

$$e^{i\theta} = b \pm i\sqrt{1-b^2}$$

$$\cos\theta = \frac{1}{2}(A+D)$$

and $\boxed{r_s = r_{max}\sin(s\theta + \delta)}$

for stability we need $\cos\theta$ real, so

$$-1 \leq \frac{1}{2}(A+D) \leq +1$$

$$-1 \leq \left(1 - \frac{d}{2f}\right) \leq +1$$

$$-1 \leq \left(\frac{d}{2f} - 1\right) \leq +1$$

$$0 \leq \frac{d}{2f} \leq 2$$

$$\boxed{0 \leq d \leq 4f} \qquad 6.2\text{-}1$$

$$\boxed{\cos\theta = \frac{1}{2}(A + D) = \left(1 - \frac{d}{2f}\right)} \quad 6.2\text{-}2$$

$$r_0 = r_{max} \sin(\delta)$$

$$r_{max} = \frac{r_0}{\sin\delta}$$

$$r_1 = r_0 + dr_0'$$

$$r_1 = r_{max} \sin(\theta + \delta)$$

$$r_0 + dr_0' = \frac{r_0}{\sin\delta} \sin(\theta + \delta)$$

$$\sin(\delta)\left(1 + \frac{dr_0'}{r_0}\right) = \sin(\theta + \delta)$$

$$\sin\delta\left(1 + \frac{dr_0'}{r_0}\right) = \sin\theta\cos\delta + \cos\theta\sin\delta$$

$$\left(1+\frac{d r_0'}{r_0}\right)=\sin\theta\,\frac{1}{\tan\delta}+\cos\theta$$

since $\cos\theta = \frac{1}{2}(A+D)$

$$\sin\theta=\sqrt{1-\cos^2\theta}$$

$$=\sqrt{1-\left[\tfrac{1}{2}(A+D)\right]^2}$$

$$\left(1+\frac{d r_0'}{r_0}\right)=\sqrt{1-\tfrac{1}{4}(A+D)^2}\,\frac{1}{\tan\delta}+\tfrac{1}{2}(A+D)$$

$$\left(1+\frac{d r_0'}{r_0}\right)-\frac{1}{2}(A+D)=\sqrt{1-\tfrac{1}{4}(A+D)^2}\,\frac{1}{\tan\delta}$$

$$\tan\delta\left[\left(1+\frac{d r_0'}{r_0}\right)-\frac{1}{2}\left(2-\frac{d}{f}\right)\right]=\sqrt{1-\tfrac{1}{4}(2-d/f)^2}$$

$$\tan\delta=\frac{\sqrt{1-\frac{1}{4}(2-d/f)^2}}{\left[\left(1+\frac{d r_0'}{r_0}\right)-\frac{1}{2}\left(2-\frac{d}{f}\right)\right]}$$

$$\tan\delta=\frac{\sqrt{4-4+4\,d/f-(d/f)^2}}{\left[2\left(1+\frac{d r_0'}{r_0}\right)-\left(2-\frac{d}{f}\right)\right]}$$

$$\tan\delta = \frac{\frac{d}{f}\sqrt{4\frac{f}{d}-1}}{2\frac{d r_0'}{r_0} + \frac{d}{f}}$$

$$\tan\delta = \frac{\sqrt{4\frac{f}{d}-1}}{2f\frac{r_0'}{r_0}+1}$$

$$r_{max} = \frac{r_0}{\sin\delta}$$

$$\sin\delta = \frac{\tan\delta}{\pm\sqrt{1+\tan^2\delta}}$$

$$r_{max} \doteq r_0\frac{\sqrt{1+\tan^2\delta}}{\tan\delta}$$

$$r_{max}^2 = r_0^2\left[\frac{1}{(\tan\delta)^2}+1\right]$$

$$r_{max}^2 = r_0^2\left[\frac{\left(2f\frac{r_0'}{r_0}+1\right)^2}{4f/d-1}+1\right]$$

$$r_{max}^2 = r_0^2\left[\frac{4f^2\left(\frac{r_0'}{r_0}\right)^2 + 4f\frac{r_0'}{r_0}+1+4\frac{f}{d}-1}{4f/d-1}\right]$$

$$r_{max}^2 = r_0^2\left[\frac{4f^2\left(\frac{r_0'}{r_0}\right)^2 + 4f\frac{r_0'}{r_0} + 4\frac{f}{d}}{4^f/d - 1}\right]$$

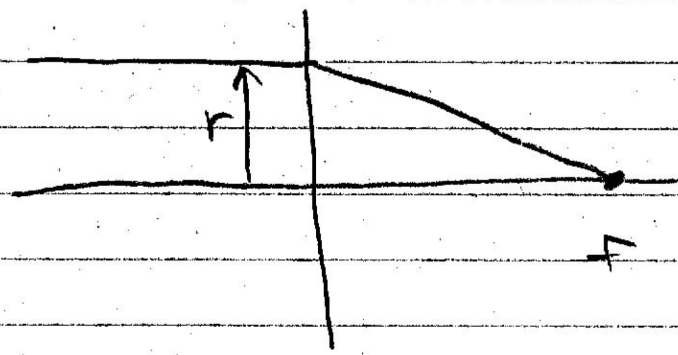
$$= r_0^2\, 4f\left[\frac{df\left(\frac{r_0'}{r_0}\right)^2 + d\frac{r_0'}{r_0} + 1}{4f-d}\right]$$

$$= \frac{(4f)\quad(df\,r_0'^2 + dr_0'r_0 + r_0^2)}{(4f-d)}$$

$$\boxed{r_{max}^2 = \frac{4f}{(4f-d)}\left(r_0^2 + dr_0r_0' + df r_0'^2\right)}$$

6.3

An ideal thin lens will image a plane wave (from an object at ∞) onto a point. It does this by making the total phase change equal for all paths, so that the light will interfere conductivly



the plane wave has the same phase at all points as it hits the lens, but the paths are longer for r larger, the path length is $\sqrt{r^2+f^2}$

and so the total phase change is

$$\Delta \phi = \frac{\sqrt{r^2+f^2}}{\lambda/n} = \frac{K}{2\pi}\sqrt{f^2+r^2}$$

So to get constructive interference the lens must provide the opposite phase shift remember $E \propto e^{-i\vec{k}\cdot\vec{r}}$

So the needed phase shift is positive

$$E_R(x,y) = E_L(x,y)\exp\left[+ik\sqrt{f^2+r^2}\right]$$

where $r^2 = x^2+y^2$

$$E_R(r) = E_L(r)\exp\left[+ik\,f\left(1+\frac{r^2}{f^2}\right)^{1/2}\right]$$

the paraxial approximation is $r \ll f$

$$E_R(r) \approx E_L(r)\exp\left[+ikf\left(1+\frac{r^2}{2f^2}+\cdots\right)\right]$$

the overall phase shift does not make a difference, so

$$E_R(r) \approx E_L(r)\exp\left[+ik\frac{r^2}{f}\right]$$

$$\approx E_L(r)\exp\left[+ik\frac{x^2+y^2}{f}\right]$$

6.6 $r_{i+1} = A r_i + B r_i'$

$r_{i+1}' = C r_i + D r_i'$

(1) Homogeneous medium of length d
the slope does not change,
so $C = 0$ & $D = 1$
If the rays position is changed, the output changes by the same amount so $A = 1$

the change in the radius is slope·distance
so $B = d$

$$\Rightarrow \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

(2) thin lens: the output is the same radius as the input, so $A = 1$ $B = 0$
a ray going through the center is not changed in its slope, so $D = 1$
a ray with $r_i' = 0$ goes through the focus so $r_{i+1}' = -\frac{r}{f}$ so $C = -\frac{1}{f}$

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

(3) $r_{i+1} = r_i$ so $A = 1$ & $B = 0$
change in slope is indep. of
r, so $C = 0$

We then have $\dfrac{\sin\theta_{i+1}}{\sin\theta_i} = \dfrac{n_1}{n_2}$

Using the paraxial approximation

$$\sin\theta \approx \theta = r'$$

$$\frac{r'_{i+1}}{r'_i} = \frac{n_1}{n_2} = D = \frac{n_1}{n_2}$$

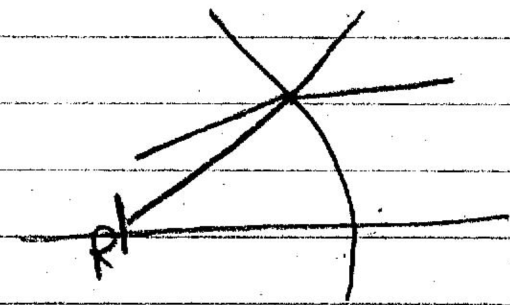
$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}$$

(4) for a curved dielectric
interface, the position does
not chang (r) so $A = 1$ $B = 0$

From Snell's law

$$\frac{\sin\theta_{i+1}}{\sin\theta_i} = \frac{n_1}{n_2}$$

but since the interface is not flat, we need to calculate θ



the radius R is $\perp$ to the interface so θ is the difference between the slope of that line and r'

$$\theta_i = r_i' - \frac{r_i}{R}$$

so we have
$$\frac{r_{i+1}' - \frac{r_i}{R}}{r_i' - \frac{r_i}{R}} = \frac{n_1}{n_2}$$

$$r_{i+1}' - \frac{r_i}{R} = \frac{n_1}{n_2}\left(r_i' - \frac{r_i}{R}\right)$$

$$r_{i+1}' = \frac{n_1}{n_2}(r_i') + \left(-\frac{n_1}{n_2} + 1\right)\frac{r_i}{R}$$

$$\Rightarrow C = \left(-\frac{n_1}{n_2} + 1\right)\frac{1}{R} = \left(\frac{n_2 - n_1}{n_1}\right)\frac{1}{R}$$

$$D = \frac{n_1}{n_2}$$

$$\begin{pmatrix} 1 & 0 \\ \frac{(n_2 - n_1)}{n_1 R} & \frac{n_1}{n_2} \end{pmatrix}$$

5) the radius does not changes so again we have

$$A = 1 \quad \& \quad B = 0$$

We now have $\Theta_i = \theta_{i+1}$ from the normal to the surface

The normal has the same slope as the radius like R

so $\theta_i = r_i' - \frac{r_o}{R}$

$$\theta_{i+1} = r_{i+1}' + \frac{r_o}{R}$$

$$r_{i+1}' + \frac{r_o}{R} = r_i' - \frac{r_o}{R}$$

$$r_{i+1}' = r_i' - 2\frac{r_o}{R}$$

$$\begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}$$

6) this is directly from 6.4-5, for a media of length l

$$r(l) = \cos\left(\sqrt{\frac{K_2}{K}}\, l\right) r_0 + \sqrt{\frac{K}{K_2}} \sin\left(\sqrt{\frac{K_2}{K}}\, l\right) r_0'$$

$$r'(l) = -\sqrt{\frac{K_2}{K}} \sin\left(\sqrt{\frac{K_2}{K}}\, l\right) r_0 + \cos\left(\sqrt{\frac{K_2}{K}}\, l\right) r_0'$$

$$\begin{pmatrix} \cos\left(\sqrt{\frac{K_2}{K}}\, l\right) & \sqrt{\frac{K}{K_2}} \sin\left(\sqrt{\frac{K_2}{K}}\, l\right) \\ -\sqrt{\frac{K_2}{K}} \sin\left(\sqrt{\frac{K_2}{K}}\, l\right) & \cos\sqrt{\frac{K_2}{K}}\, l \end{pmatrix}$$

6.8(a)

the ABCD matrix
to a distance $\ell_3 > \ell_2$
from the waist position is

$$\begin{pmatrix} 1 & \ell_3-\ell_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}\begin{pmatrix} 1 & \ell_2-\ell_1 \\ 0 & 1 \end{pmatrix}$$

$$\cdot\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\begin{pmatrix} 1 & \ell_1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \ell_3-\ell_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \ell_2-\ell_1 \\ 0 & 1/n \end{pmatrix}\begin{pmatrix} 1 & \ell_1 \\ 0 & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \ell_3-\ell_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \ell_1+(\ell_2-\ell_1)n \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \ell_1+(\ell_2-\ell_1)n_1+\ell_3-\ell_2 \\ 0 & 1 \end{pmatrix}$$

Since the propagation is equivalent
to propagation by some extra distance
the far field angle is not changed.

(b) One way to do this is to start at the beam waist, propagate backward by a distance l_1, and then forward again through the dielectric interface and then the distance to the new waist, following example in 6.7

at the original waist we have

$$\frac{1}{q_1} = \frac{1}{R_1} - i\frac{\lambda}{\pi w_{01}^2 n} = -i\frac{\lambda}{\pi w_0^2}$$

From 6.7-6 and $\begin{pmatrix} 1 & -l_1 \\ 0 & 1 \end{pmatrix}$

we have

$$q(-l_1) = \frac{q_1 - l_1}{1}$$

when we propagate into the material with index n we get

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$$

$$q_+(\ \ ) = (q_1 - l_1)\,n$$

$$q_+ = i\frac{\pi w_0^2 n}{\lambda} - l n$$

propagating forward a distance
L to the new waist, we see

$$q_{\text{new waist}} = i\frac{\pi w_0^2}{\lambda} n - l n + L$$

so the distance to the new waist
is $L = n l_1$

and at that point

$$q_{\text{new waist}} = i\frac{\pi w_0^2}{\lambda} n$$

$$\frac{1}{q_{\text{new waist}}} = -i\frac{\lambda}{\pi \underset{\substack{\uparrow \\ \text{new waist}}}{w_0^2} n}$$

So the new waist is the
same as the old waist