EE2J6
Homework 9
Solution

1) The number of downward transitions is proportional to N_2 and $(n+1)$" where n is the number of photons in the mode.

The number of upward transition is proportional to N_1 and n

To have
absorption rate = emission rate

$$\frac{N_2(n+1)}{N_1(n)} = 1$$

$$\frac{N_2}{N_1}\left(1+\frac{1}{n}\right) = 1$$

$$\left(1+\frac{1}{n}\right) = \frac{N_1}{N_2}$$

$$\frac{1}{n} = \frac{N_1}{N_2} - 1$$

$$n = \left(\frac{N_1}{N_2} - 1\right)^{-1}$$

We can then use the Blackbody equation, substituting this for the number of photons in a mode (5.7-5)

$$\rho(\nu) = \frac{8\pi h n^2 \nu^3}{c^3}\left(\frac{N_1}{N_2} - 1\right)^{-1}$$

as $N_2 \to N_1$ this goes to infinity

2) We will still have

$$(W_{21}')_i = B_{21}\,\rho(\nu)$$
$$(W_{12}')_i = B_{12}\,\rho(\nu)$$

$$W_{21}' = B_{21}\,\rho(\nu) + A$$
$$W_{12}' = B_{12}\,\rho(\nu)$$

but $\rho(\nu)$ will be an energy density per unit area for a particular cavity with d

We now need to calculate the density of modes per unit area the area of K-space for $|\vec{K}|<K$ is $\pi |K|^2$

and the average area per mode is $\left(\frac{2\pi}{L}\right)^2$

where L is the Length of the cavity in the two "long" dimentions

we also have 2 polarizations per mode

$$N(K) = 2\,\frac{\pi |K|^2}{\left(\frac{2\pi}{L}\right)^2}$$

$$N(K) = \frac{\pi K^2 L^2}{2\pi^2} = \frac{K^2 A}{2\pi} \qquad A\text{-Area}$$

$$K = 2\pi\nu\, n/c$$

$$\frac{N(\nu)}{A} = \frac{1}{2\pi}\left(2\pi\nu\, n/c\right)^2$$

$$\frac{N(\nu)}{A} = 2\pi\left(\frac{n}{c}\right)^2 \nu^2$$

$$P_{area}(\nu) = 4\pi\left(\frac{n}{c}\right)^2 \nu$$

we then have an energy density per unit area and frequency

$$\rho_{area}(\nu) = 4\pi\left(\frac{n}{c}\right)^2 \nu \left(\frac{h\nu}{e^{(h\nu/kT)}-1}\right)$$

we then have

$$N_2 W'_{21} = N_1 (W'_{12})$$

$$N_2 \left(B_{21}\, \rho_A(\nu) + A\right) = N_1\, B_{12}\, \rho_A(\nu)$$

where $\dfrac{N_2}{N_1} = \dfrac{g_2}{g_1} e^{-h\nu/kT}$

and plugging in $\rho_A(\nu)$

$$\frac{g_2}{g_1} e^{-h\nu/kT}\left(B_{21}\, 4\pi \left(\frac{n}{c}\right)^2 h\nu^2 \left(\frac{1}{e^{h\nu/kT}-1}\right) + A\right.$$

$$= B_{12}\, 4\pi \left(\frac{n}{c}\right)^2 h\nu^2 \left(\frac{1}{e^{h\nu/kT}-1}\right)$$

$$\frac{4\pi \left(\frac{n}{c}\right)^2 h\nu^2}{\left(e^{h\nu/kT}-1\right)} = \frac{A\ (g_2/g_1)}{B_{12}\, e^{h\nu/kT} - B_{21}(g_2/g_1)}$$

$$\Rightarrow B_{12} = B_{21} \frac{g_2}{g_1}$$

$$\frac{A}{B_{21}} = 4\pi \left(\frac{n}{c}\right)^2 h\nu^2$$

3)

$$\Delta \nu_D = 2\nu_0 \sqrt{\frac{2kT}{Mc^2} \ln 2}$$

$$M_{Ne} = 20 \times 1.66 \times 10^{-27} \quad (Kg)$$

$$\Delta \nu_D \sim 2 \times 10^8 \text{ Hz}$$

For the common 6328Å (red) line

this is inhomogeneous broadening