EE 236
9/1/04
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Representations, orthogonality

In the first lecture, we defined Kets, Bras and Operators as abstract symbols for dealing with QM systems. Today, we will look at some specific ways of looking at QM problems with a countable number of states

(Example of a QM system with an uncountable number of states: particle in free space
$\psi(\vec{r})$

lets say that there happens to be a set of states such that

$$|\psi\rangle = A_1|\phi_1\rangle + A_2|\phi_2\rangle + A_3 \cdots$$

for some $A_1, A_2 \cdots$

We then say that the space of all states is <u>spanned</u> by the set $|\phi_1\rangle$, $|\phi_2\rangle$ ---
and that they form a <u>complete</u> set.

If the set also has the property

$$\langle \phi_m | \phi_n \rangle = 0 \quad m \neq n$$

then the set is said to be <u>orthogonal</u>

If the elements also have the property

$$\langle \phi_n | \phi_n \rangle = 1 \quad \text{for all } n$$

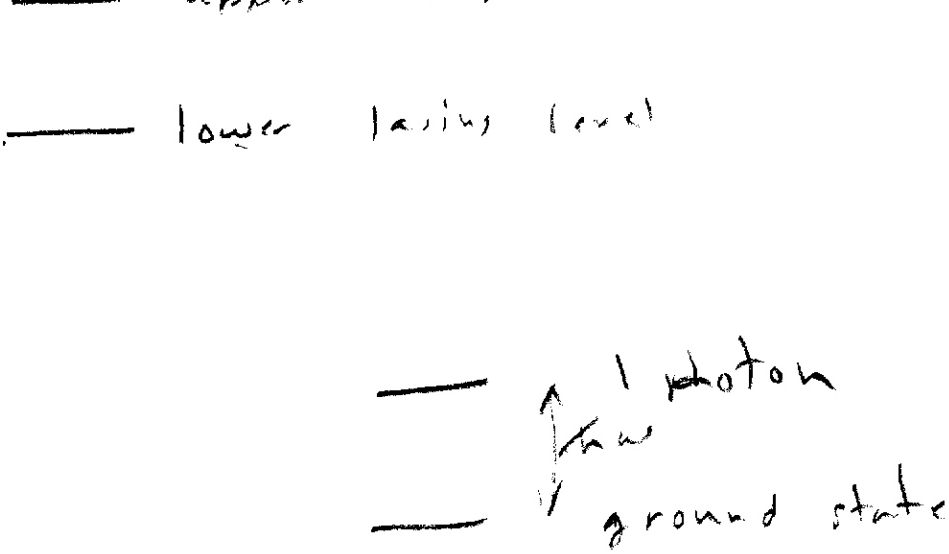
then the set is normalized

If a complete orthogonal set of M states spans a space of a QM system, it is said to be an M-state system.

Example. If we consider only the spin of an electron, then it is a two-state system

$|up\rangle$ $|down\rangle$

Often, we can approximate a system with many states by limiting the system to just a few of the possibilities. An atom might be modeled using just the "lasing levels" and the photons might be modeled by a ground state & the state with one photon in one mode.



If we were to somehow construct a complete, orthogonal set of normalized eigenvectors, we can then use them to translate any problem with states and operators into vectors and matrices

$$|\psi\rangle = \sum_n \psi_n |\phi_n\rangle$$

$$\langle \phi_m | \psi \rangle = \sum_n \psi_n \langle \phi_m | \phi_n \rangle$$

$$\langle \phi_m | \psi \rangle = \psi_m$$

So a general state of the system can be written

$$|\psi\rangle = \sum_m \langle \phi_m | \psi \rangle |\phi_m\rangle$$

lets rewrite this as

$$|\psi\rangle = \sum_n |\phi_n\rangle \langle \phi_n | \psi \rangle$$

we can now identify an expression for an identity operator

$$I = \sum_n |\phi_n\rangle\langle \phi_n|$$

and we can use that Identity operator to find a matrix representation for any operator

$$\hat{A}$$

$$= \sum_m \sum_n |\phi_m\rangle\langle\phi_m| \hat{A} |\phi_n\rangle\langle\phi_n|$$

$$= \sum_m \sum_n A_{mn} |\phi_m\rangle\langle\phi_n|$$

Notice that the vectors φ_m and the numbers A_{mn} depend on the <u>representation</u> i.e. exactly which complete, orthogonal set is used to represent them.

Lets look at this from the opposite point of view. If a Q.M. problem can be spanned by a complete, orthogonal set, then any state of the system corresponds to a point in an n dimensional linear space, and therefore be

represented by a vector, and any Operator corresponds to a linear transformation and can be represented by a matrix.

The results we will obtain are more general, but lets look at just finite linear spaces.

In a Hilbert space, for any operator there are specific states which have the property that

$$\hat{A} \, |A_n\rangle = a_n \, |A_n\rangle$$

where a_n is a complex number. For the case of a finite linear space, this is just the corresponding result for matrices, resulting from any representation

These specific states are called the eigenstates of the operator $\hat{A}$ the values a_n are the eigenvalues corresponding to those eigenstates

If we interact with a system or "measure" it, the system in general will change. If we use the operator $\hat{A}$ to represent the change,

If we demand that our measurement be repeatable, and that it tells us something about the state of the system, we will effectively be acting with the operator more than once

$\hat{A}\,|\psi\rangle$ measure once

$\hat{A}\hat{A}\,|\psi\rangle$ measure again

$\hat{A}\hat{A}\hat{A}\,|\psi\rangle$ measure a third time

We will look at "measurement" more closely later, but for now lets recognise that for a "measurement" to be repeatable, the action of a "measurement" must produce an eigenstate of $\hat{A} \rightarrow |A_n\rangle$ so that further "measurements" don't produce different results each time

If we look at the Hermitian operators, they have the properties which will be nice in this regard, so we might state the theory of "measurement":

→ All observables correspond to Hermitian operators

→ Results of observations are the eigenvalues of these operators

→ After a "measurement" the observed system is in the eigenstate corresponding to the observed eigenstate

The eigenvalues of a Hermitian operator can be shown to be real:

$$\hat{A}\,|\phi_n\rangle = a_n|\phi_n\rangle \quad \#1$$
$$\hat{A}\,|\phi_m\rangle = a_m|\phi_m\rangle \quad \#2$$

because $\hat{A}$ is hermitian

$$\langle\phi_m|\hat{A} = a_m^*\langle\phi_m| \quad \#3$$
$$\langle\phi_n|\hat{A} = a_n^*\langle\phi_n| \quad \#4$$

we now project #1 on $\langle \phi_m |$

and use #3 to project $| \phi_n \rangle$
we get

$$\langle \phi_m | \hat{A} | \phi_n \rangle = a_n \langle \phi_m | \phi_n \rangle$$

$$\langle \phi_m | \hat{A} | \phi_n \rangle = a_m^* \langle \phi_m | \phi_n \rangle$$

subtracting, we get

$$0 = (a_n - a_m^*) \langle \phi_m | \phi_n \rangle$$

if $n = m$ a_n is real $(a_n = a_n^*)$
if $n \neq m$ <u>and</u> $a_n \neq a_m^*$ then

$\langle \phi_m | \phi_n \rangle = 0$ (they are orthogonal)

So the set of all eigenstates form a set of orthogonal states which we can project any state of the system on & "represent" it

is see they complete?

they don't have to be → we must use a set of operators which span the space → so that observing the observables describes all aspects of the D.M. system in a set of states

If an operator which spans the space has N eigenstates, we call that an N state system (2-state system, etc.)

If the eigenvectors of an operator $\hat{A}$ are complete, then <u>any</u> ket in the space can be expressed

$$|\psi\rangle = \sum_n c_n |a_n\rangle$$

where the c_n's are complex numbers

→ that is, that any state ψ can be completely represented by an ordered vector of complex numbers (a column matrix)

& therefore any linear operator can be represented by a matrix

Any observable will result in a complete set of orthogonal eigenvalues and can be used to represent the same system with <u>different</u> vectors and different matrices.

lets see how to calculate the c_n's

$$|\psi\rangle = \sum_n c_n |a_n\rangle$$

$$\langle a_n|\psi\rangle = \sum_n \langle a_n|c_n|a_n\rangle$$

$$\langle a_n|\psi\rangle = \sum_n c_n \langle a_n|a_n\rangle = c_n$$

$$c_n = \langle a_n|\psi\rangle$$

$$|\psi\rangle = \sum_n \underbrace{\langle a_n|\psi\rangle}_{\text{just a number}} |a_n\rangle$$

$$|\psi\rangle = \underbrace{\left[\sum_n |a_n\rangle\langle a_n|\right]}_{\text{Identity operator}} |\psi\rangle$$

project over the a_n's & then reexpand over them

$\Rightarrow$ we can use these to express arbitrary operators "$\hat{B}$" as matrices in the "$\hat{A}$" representation

$$|\phi\rangle = \hat{B}|\psi\rangle$$

$$|\phi\rangle = \sum_n |a_n\rangle\langle a_n|\hat{B}\sum_m |a_m\rangle\langle a_m|\psi\rangle$$