EE 236
10/1/04
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The Kramers-Kronig relationship

When an Electromagnetic field acts on a population of atoms (matter) it produces a polarization, $\vec{P}$.

In general, the polarization is a function of the Electric field locally, but not instantly $\Rightarrow$ It depends on the history of the Electric fields in the past.

If the relationship between the electric field and the polarization is linear, i.e.

$$P\{\vec{E}_1(t) + \vec{E}_2(t)\} = P\{E_1(t)\} + P\{E_2(t)\}$$

for any electric fields $\vec{E}_1(t)$ + $\vec{E}_2(t)$, we can find the response $\vec{P}(t)$ to an impulse $\vec{E}(t) = \vec{E}_0\,\delta(t-t')$ and then compose any $\vec{E}(t)$ as a sum over these delta functions. We get a Green's function for $\vec{P}$

the equation $P = \int_0^\infty G(\tau)\, E(\vec{r}, t-\tau)\, dt$
(for a time invariant, linear + causal relationship)
— is a special case, when transformed
as discussed earlier, we have

$$\mathcal{E}(\omega) = \mathcal{E}_0 + \int_0^\infty G(\tau)\, e^{i\omega\tau}\, d\tau$$

and so we have

$$\vec{D}_\omega = \mathcal{E}_\omega \vec{E}$$

→ $\mathcal{E}_\omega$ is now restricted to the isotropic, not optically active, case so it is a complex scaler

The Kramers-Kronig relations are a consequence of these approximations. Yariv treats this in appendix I, or better, see Jackson 3rd ed pp 332

This follows from the realization that $\mathcal{E}(\omega)$ is a complex variable of the complex variable ω, and is is an analytic function of ω in the upper half plane

We also need the requirement $G(\tau) \to 0$ as $\tau \to \infty$, so the material has no "memory"

Note that the equation

$$\vec{P} = \int_0^\infty G(\tau)\, E(\vec{r}, t-\tau)\, d\tau$$

is not in phasor notation $\Rightarrow$ $G(\tau)$ must be a real function ($\vec{E}$ & $\vec{P}$ are real)

So when we look at

$$\mathcal{E} = \mathcal{E}_0 + \int_0^\infty G(\tau) e^{i\omega t}\, d\tau$$

$$\mathcal{E}^* = \mathcal{E}_0 + \int_0^\infty G(\tau) e^{-i\omega t}\, d\tau$$

$$\Rightarrow \mathcal{E}(\omega) = \mathcal{E}^*(\omega^*)$$

Note that a pole in the upper half plane corresponds to a complex frequency

$$\omega = \omega_r + i\gamma$$

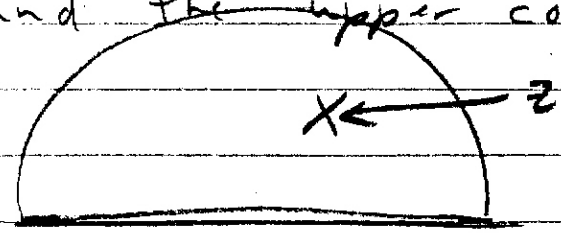
where there is a potentially ∞ $\vec{P}$ in the presence of a decaying $\vec{E}$ field

$$\vec{E} = E(r)\, e^{i(\omega_r + i\gamma)t}$$

$$= E(r)\, e^{i\omega_r - \gamma t}$$

This is all simply a consequence of the fact that what we see as two very different things — (gain or loss) vs index of refraction, are due to the same physical process

Now consider the integral along a contour around the upper complex plane:



$$\int_{-\infty}^{\infty} \frac{\mathcal{E}(\omega') - \mathcal{E}_0}{\omega' - z}\, d\omega'$$

because this is an analytic expression except for the pole at $\omega' - z$, the result of the contour integration is the residue at the pole

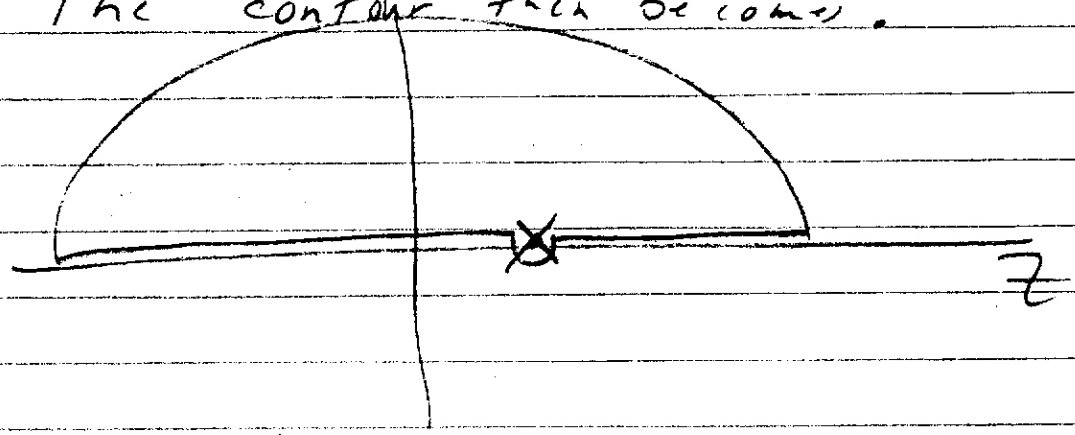
$$\int_{-\infty}^{\infty} \frac{\mathcal{E}(\omega') - \mathcal{E}_0}{\omega' - z}\, d\omega' = 2\pi i \left[\mathcal{E}(z) - \mathcal{E}_0\right]$$

So we get

$$\mathcal{E}(z) = \mathcal{E}_0 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\left[\mathcal{E}(\omega')/\mathcal{E}_0 - 1\right]}{\omega' - z}\, d\omega'$$

now this expression is valid anywhere in the upper half plane, so we take the limit as the pole approaches the axis where z ($\sim \omega$) is real

The contour then becomes:



This integral has three parts, an integral along the real axis aproaching the pole symmetrically, the infantesimal integral around the bottom of the pole, and the semicircular integral which approaches infinity.

The semicircular integral at infinity goes to zero on physical grounds (Matt cannot respond infinitly fast) (the polarization responses to Gamma rays is very small)

The integral along the real axis becomes a Principal value integral from $-\infty$ to $+\infty$, and the integral around the infantesimal contour can be evaluated to 1/2 the residue of the pole $\rightarrow$

$$\varepsilon(\omega) = \varepsilon_0 + \frac{1}{\pi i} P.V. \int_{-\infty}^{\infty} \frac{[\varepsilon(z) - \varepsilon_0]}{z - \omega} dz$$

where z is now constrained to be on the real axis

Notice that the i in the denominator means that the integral of the <u>real</u> part of ε allows us to find the <u>immaginary</u> part of ε, and the integral of the <u>immaginary</u> part of ε allows us to find the real part of ε.

$$Re\{\varepsilon(\omega)\} = \varepsilon_0 + \frac{2}{\pi} P.V. \int_{-\infty}^{\infty} \frac{Im\{\varepsilon(z)\}}{z - \omega} dz$$

$$Im\{\varepsilon(\omega)\} = \frac{2\omega}{\pi} P.V. \int \frac{Re\{\varepsilon(z) - \varepsilon_0\}}{z - \omega} dz$$

multiplying the integrals by

$$\frac{(z+\omega)}{(z+\omega)}$$

we get

$$Re\{\varepsilon(\omega)\} = 1 + \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{(z+\omega)\, Im(\varepsilon(z))}{z^2-\omega^2} dz$$

$$Im\{\varepsilon(\omega)\} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(z+\omega)\left[Re\{\varepsilon(z)\} - \varepsilon_0\right]}{z^2-\omega^2} dz$$

now the integrals are expressed in terms of functions which are even $[Re(\varepsilon(z))]$ $(z^2-\omega^2)$, or odd $[Im(\varepsilon(z)), z]$

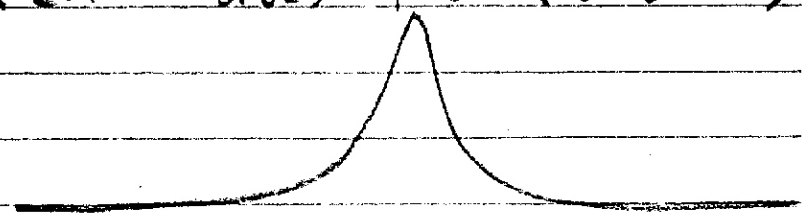
so we can write the integrals

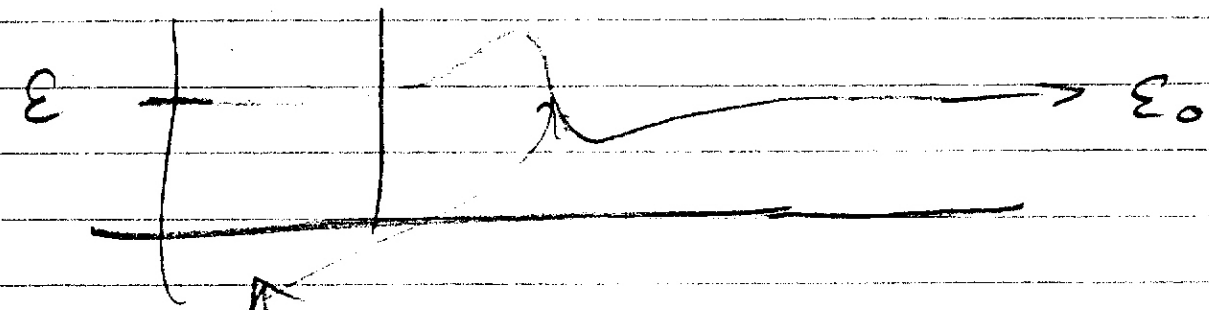
$$Re\{\varepsilon(\omega)\} = 1 + \frac{2}{\pi} P.V. \int_0^{\infty} \frac{z\, Im(\varepsilon(z))}{z^2-\omega^2} dz$$

+

$$Im\{\varepsilon(\omega)\} = -\frac{2\omega}{\pi} P.V. \int_0^{\infty} \frac{Re\{\varepsilon(z)-\varepsilon_0\}}{z^2-\omega^2} dz$$

thus if we have a material which is strongly absorbing over a narrow range of frequencies, we get an dielectric constant which does the following:





in this range, ε is approximatly independant of frequency

this region is called anomalous <u>dispersion</u>