EE C245 – ME C218
Introduction to MEMS Design
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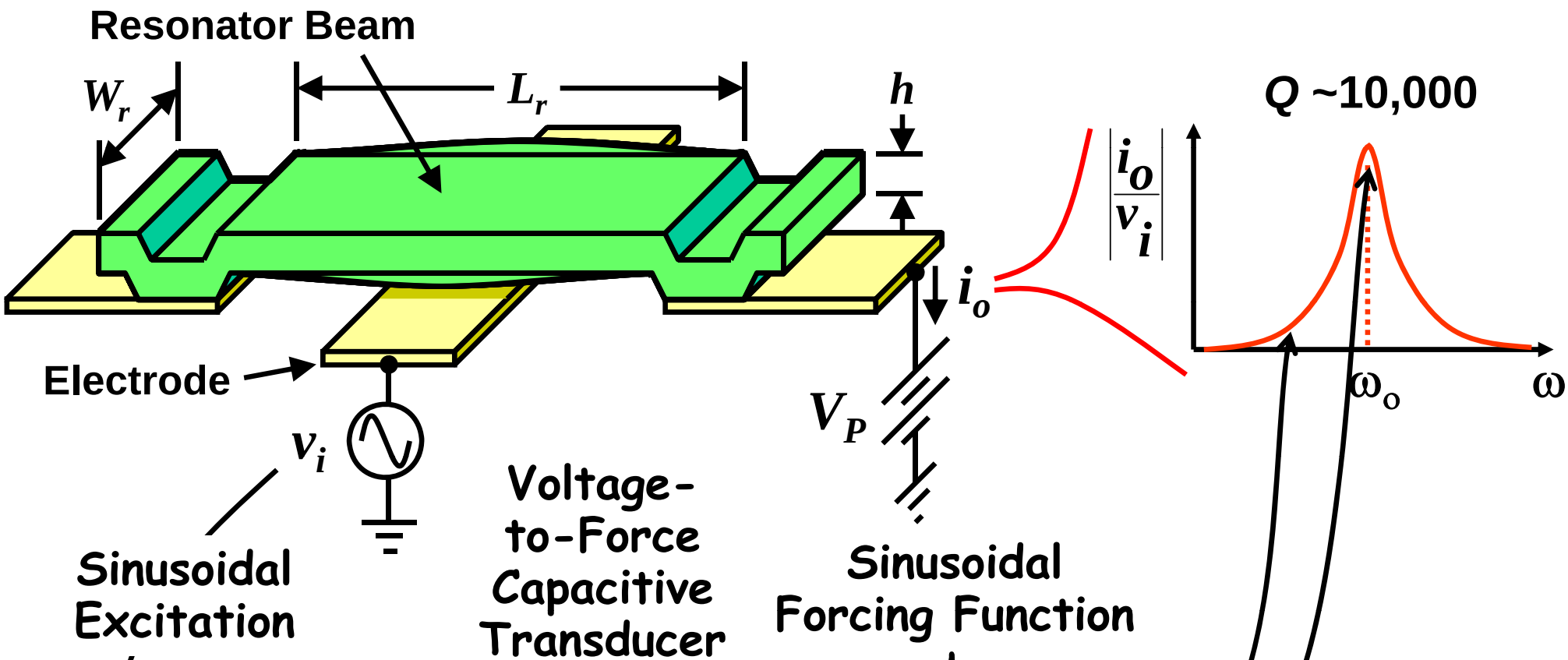
Lecture 18: Resonance Frequency

Lecture Outline

- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
    - Virtual Work
    - Energy Formulations
    - Tapered Beam Example
    - Estimating Resonance Frequency

Estimating Resonance Frequency

Clamped-Clamped Beam μResonator



Sinusoidal Excitation

Voltage-to-Force Capacitive Transducer

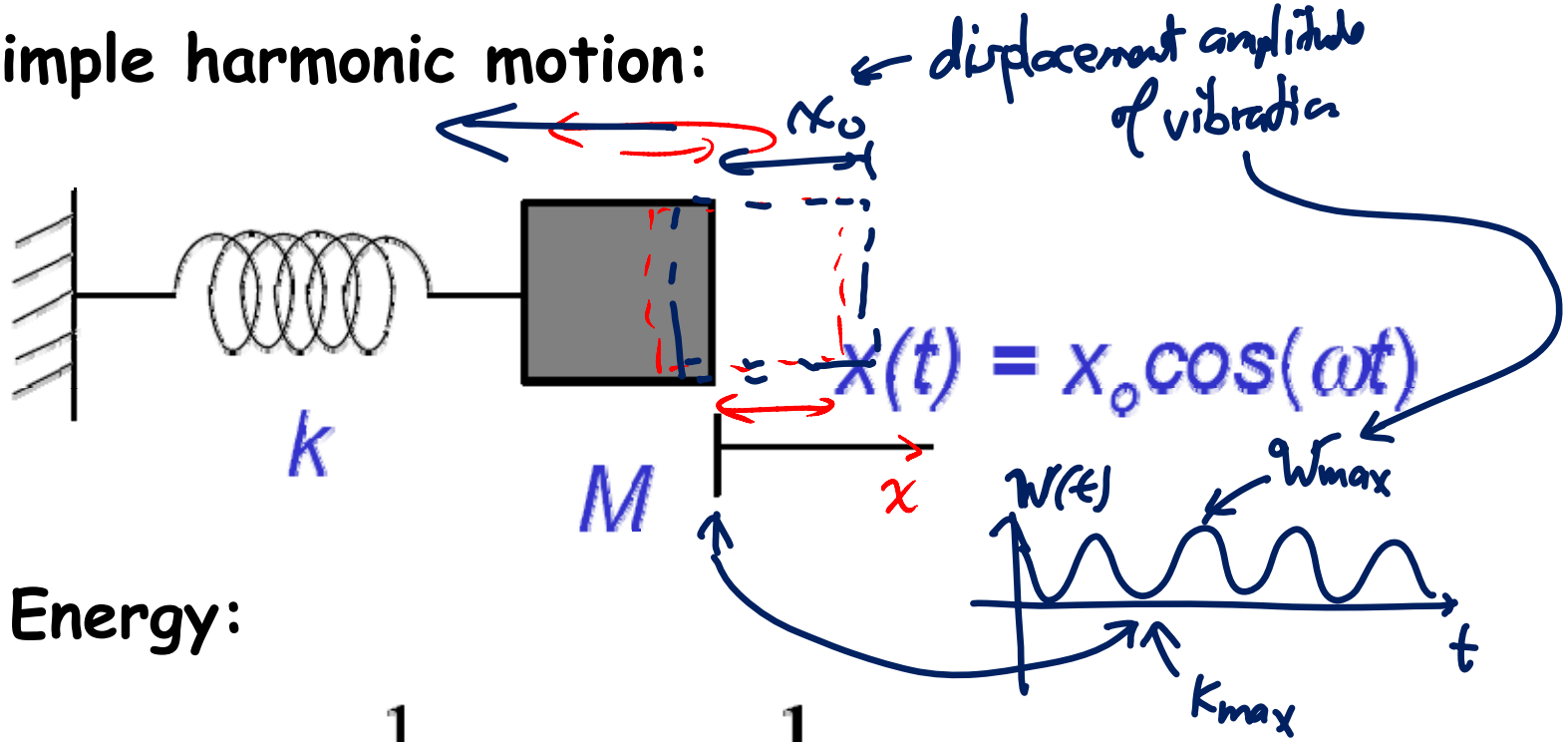
Sinusoidal Forcing Function

$$v_i = V_i \cos[\omega_o t] \longrightarrow f_i = \overbrace{F_i \cos[\omega_o t]}$$

- $\omega \neq \omega_o$: small amplitude
- $\omega = \omega_o$: maximum amplitude → beam reaches its maximum potential and kinetic energies

Estimating Resonance Frequency

- Assume simple harmonic motion:



- Potential Energy:

$$W(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kx_o^{\,2}\cos^2(\omega t)$$

- Kinetic Energy:

$$K(t) = \frac{1}{2}M\dot{x}^2(t) = \frac{1}{2}Mx_o^{\,2}\omega^2\sin^2(\omega t)$$

$\frac{dx}{dt}$ velocity

Estimating Resonance Frequency (cont)

- Energy must be conserved:
  - Potential Energy + Kinetic Energy = Total Energy
  - Must be true at every point on the mechanical structure

Occurs at peak displacement

Occurs when the beam moves through zero displacement

$$W_{max} = \frac{1}{2} k x_o^2 = K_{max} = \frac{1}{2} M \omega^2 x_o^2$$

v^2 ← velocity

Maximum Potential Energy

Stiffness

Displacement Amplitude

Maximum Kinetic Energy

Mass

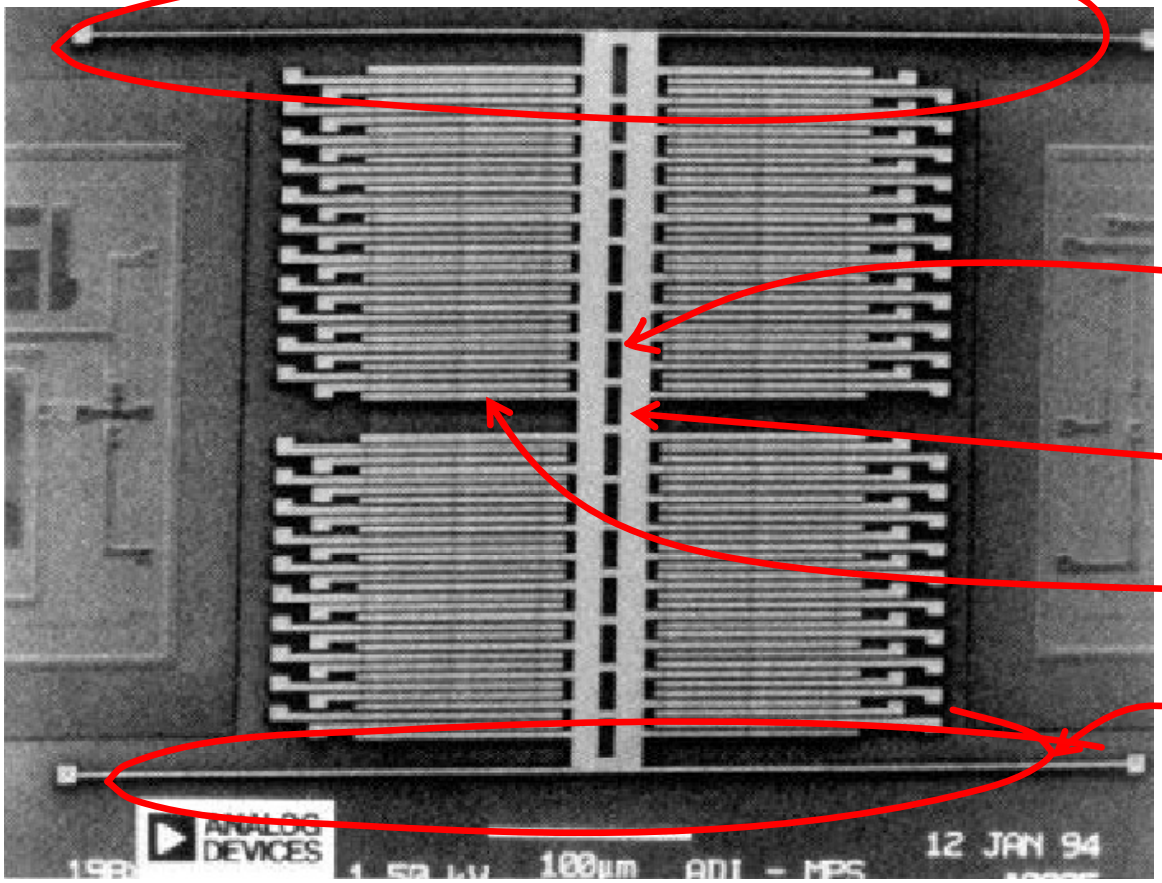
Radian Frequency

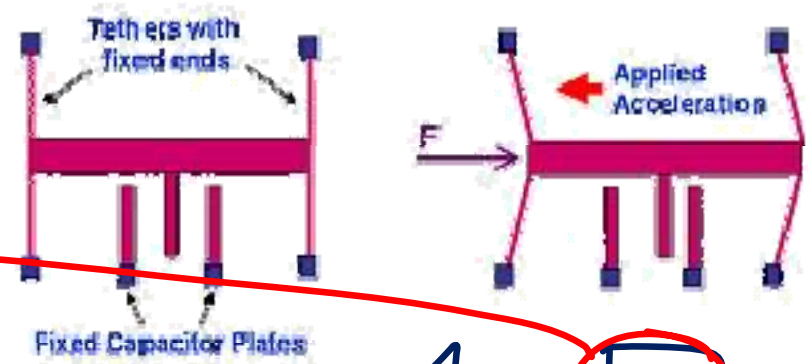
- Solving, we obtain for resonance frequency:

$$\boxed{\omega = \sqrt{\frac{k}{M}}}$$

Example: ADXL-50

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm





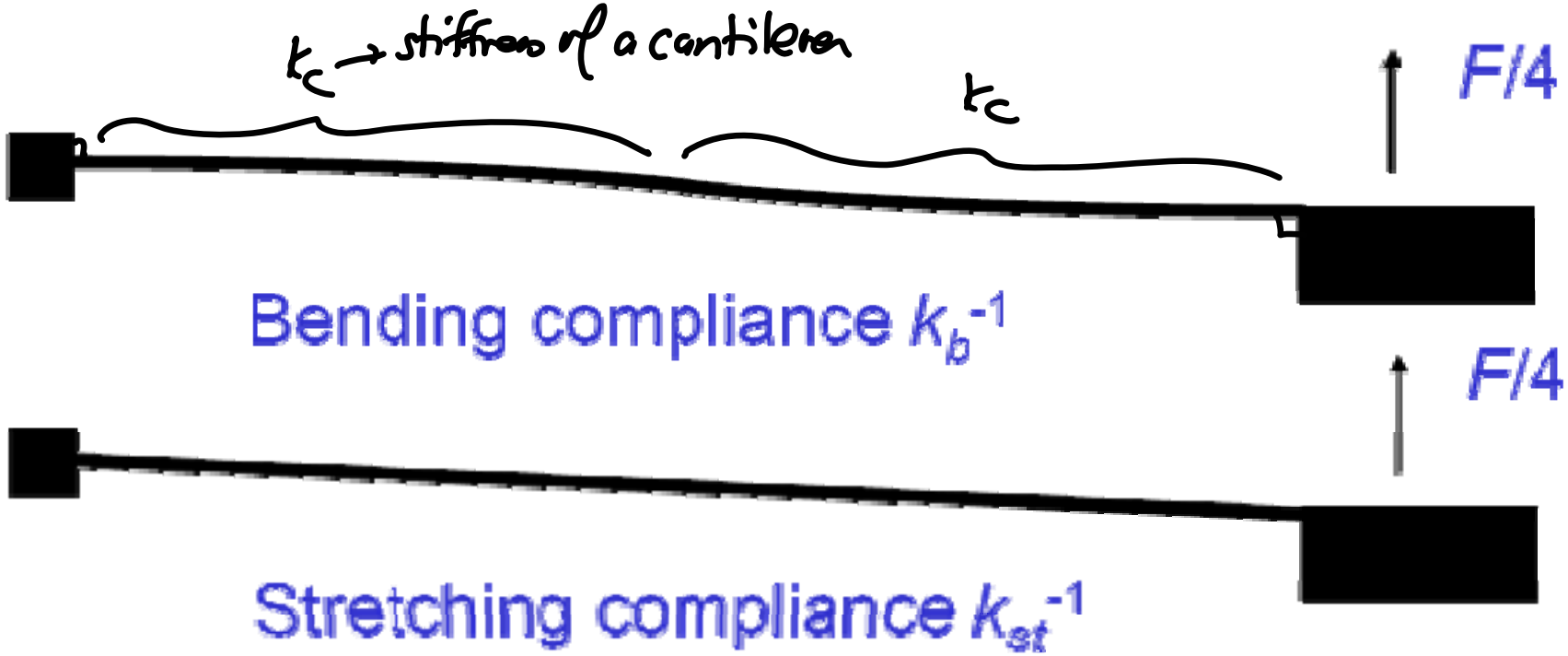
Proof Mass

Sense Finger

Suspension Beam in Tension

Lumped Spring-Mass Approximation

- Mass is dominated by the proof mass
  - 60% of mass from sense fingers
  - Mass = M = 162 ng (nano-grams)
- Suspension: four tensioned beams
  - Include both bending and stretching terms [A.P. Pisano, BSAC Inertial Sensor Short Courses, 1995-1998]



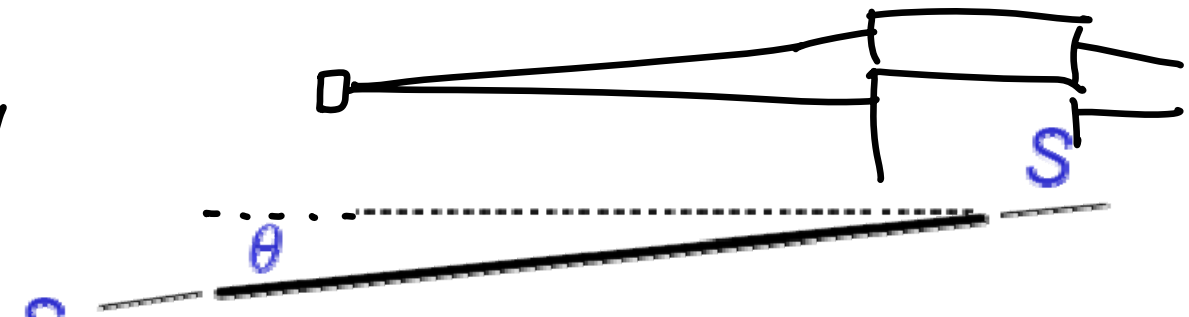
ADXL-50 Suspension Model

- Bending contribution:

$$k_b^{-1} = \left(\frac{1}{k_c} + \frac{1}{k_o}\right) = \frac{L^3}{EWh^3} = 4.2\ \mu m/\mu N$$

- Stretching contribution:

$$k_{st}^{-1} = \frac{L}{S} = \frac{L}{\sigma_r W h} = 1.14\ \mu m/\mu N$$



$$F_y = S\sin\theta \approx S(x/L) = \underbrace{\left(\frac{S}{L}\right)}_{= k_{st}} x$$

- Total spring constant: add bending to stretching

(stiffnesses are in parallel) + (4 beams in parallel)

$$k = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.48\ \mu N/\mu m$$

ADXL-50 Resonance Frequency

- Using a lumped mass-spring approximation:

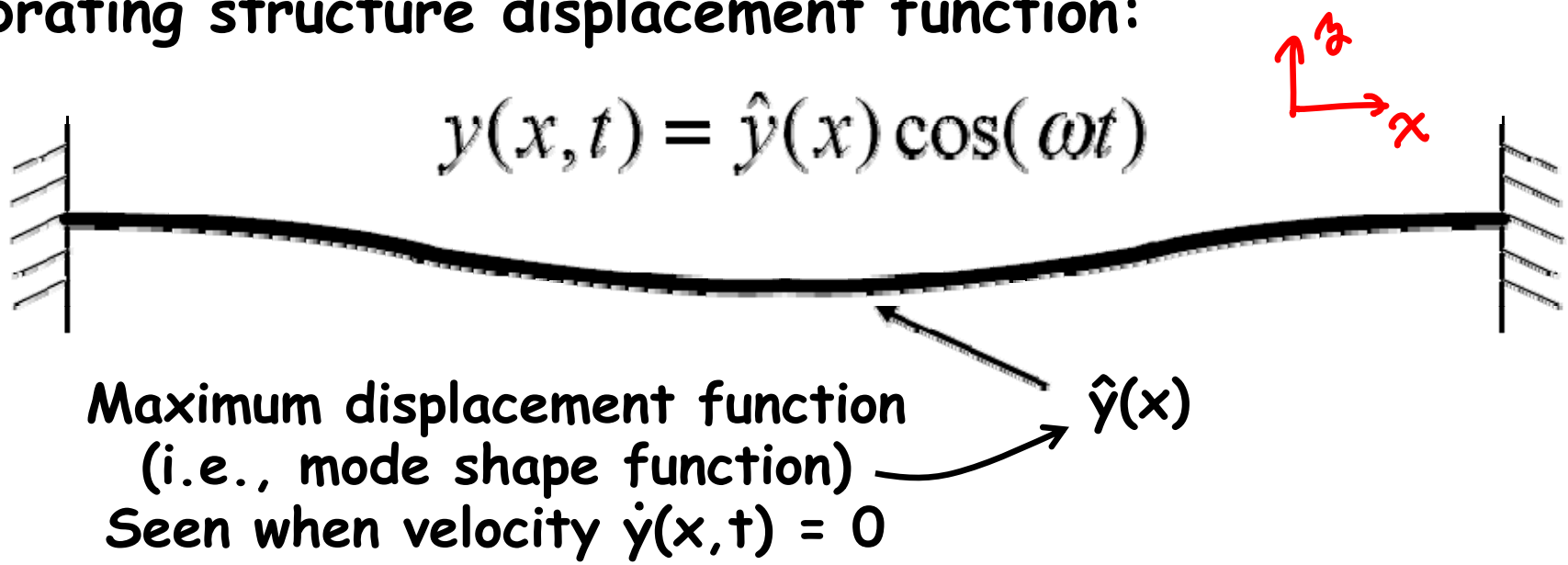
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{M}} = \frac{1}{2\pi}\sqrt{\frac{4.48N/m}{162x10^{-12}kg}} = 26.5kHz$$

- On the ADXL-50 Data Sheet: f_o = 24 kHz
  - Why the 10% difference?
  - Well, it's ~~approximate~~ … plus …
  - Above analysis does not include the frequency-pulling effect of the DC bias voltage across the plate sense fingers and stationary sense fingers … something we'll cover later on …

Distributed Mechanical Structures

- Vibrating structure displacement function:

$$y(x,t) = \hat{y}(x)\cos(\omega t)$$

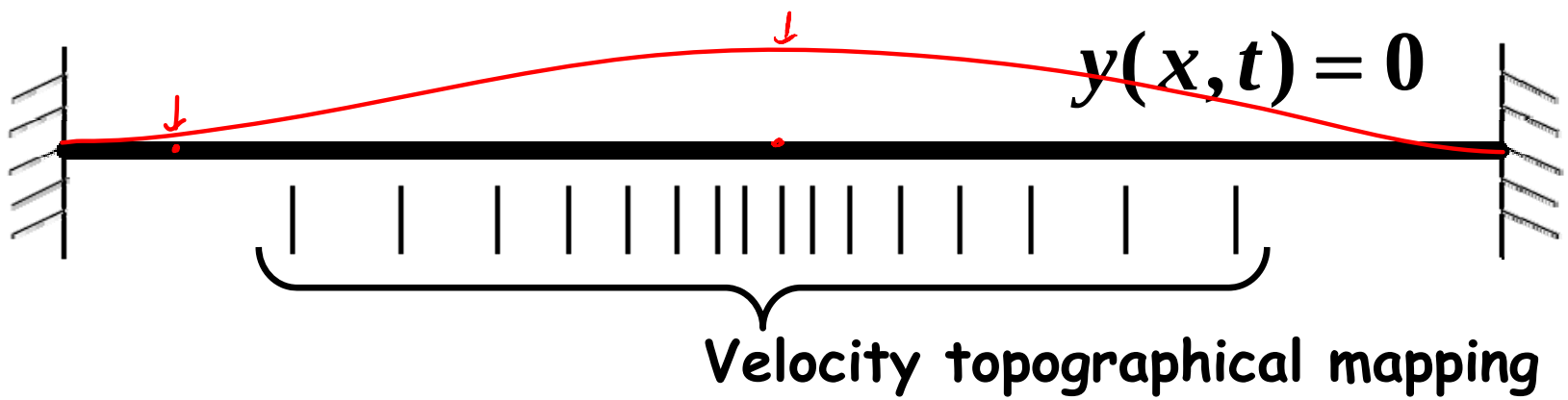


Maximum displacement function
(i.e., mode shape function)
Seen when velocity $\dot{y}(x,t) = 0$

- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy $\mathcal{W}_{max}$ at the point of maximum displacement (e.g., when t=0, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

Maximum Kinetic Energy

- Displacement: $y(x,t) = \hat{y}(x)\cos[\omega t]$
- Velocity: $v(x,t) = \dfrac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin[\omega t]$
- At times t = π/(2ω), 3π/(2ω), …

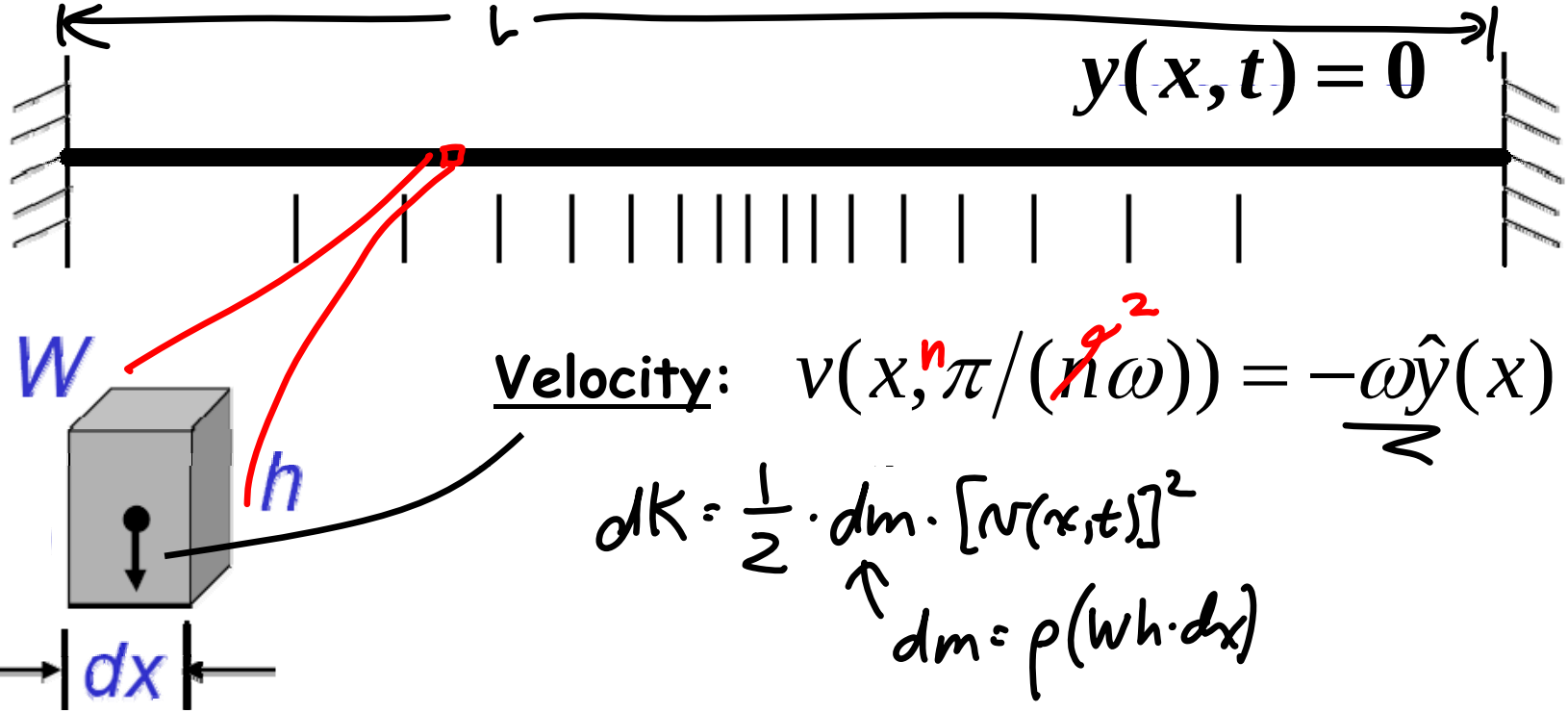


- The displacement of the structure is y(x,t) = 0
- The velocity is maximum and all of the energy in the structure is kinetic (since W=0):

$$v(x, \overset{n}{\cancel{n}}\pi/(\overset{2}{\cancel{n}}\omega)) = -\omega\hat{y}(x)$$

Maximum Kinetic Energy (cont)

- At times $t = \pi/(2\omega)$, $3\pi/(2\omega)$, …



Velocity: $v(x, n\pi/(\cancel{n}\omega)) = -\omega\hat{y}(x)$

$dK = \frac{1}{2} \cdot dm \cdot [v(x,t)]^2$

$dm = \rho(Wh \cdot dx)$

⇒ Maximum Kinetic Energy:

$$K_{max} = \int_0^L \frac{1}{2}\rho Wh\,dx\, v^2(x,t') = \int_0^L \frac{1}{2}\rho Wh\omega^2 \hat{y}^2(x)\,dx$$

time when velocity is maximum

The Raleigh-Ritz Method

- Equate the maximum potential and maximum kinetic energies:

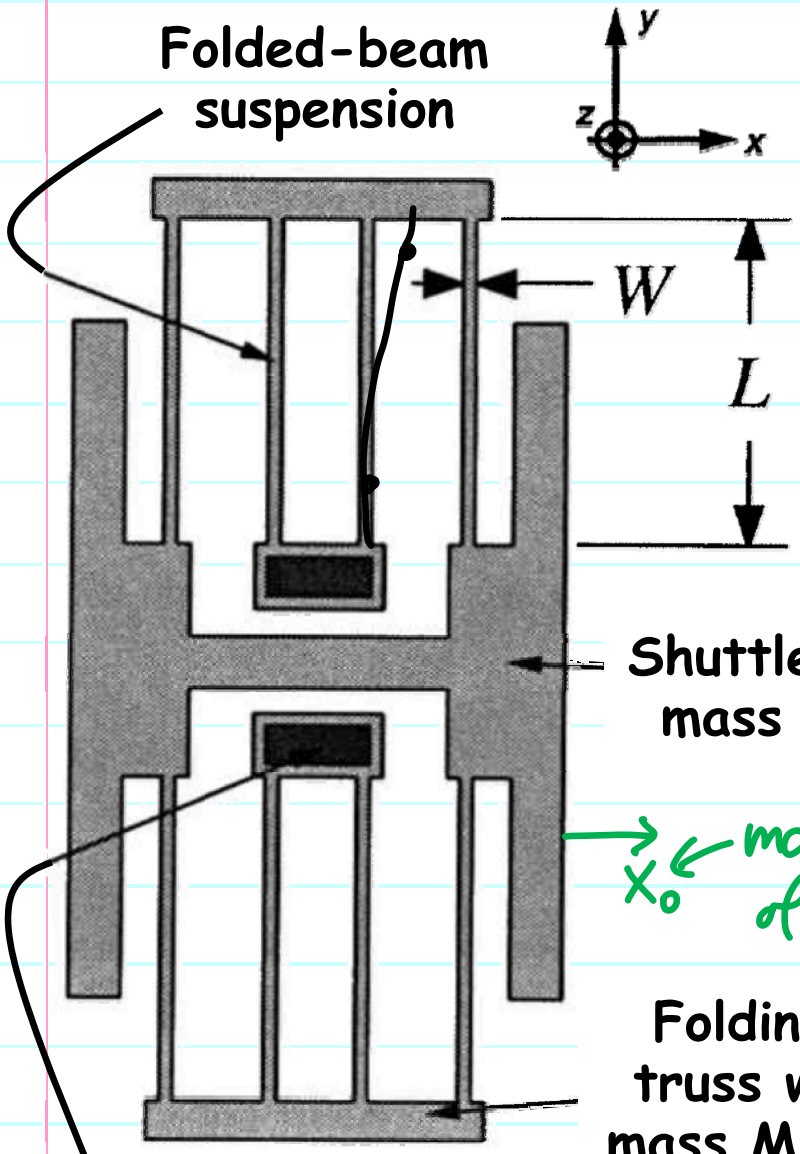
$$K_{\max} = \int_0^L \frac{1}{2}\rho W h \omega^2 \hat{y}^2(x)dx = \mathcal{W}_{\max}$$

- Rearranging yields for resonance frequency:

$$\boxed{\omega = \sqrt{\frac{\mathcal{W}_{\max}}{\int_0^L \frac{1}{2}\rho W h \hat{y}^2(x)dx}}}$$

ω = resonance frequency
$\mathcal{W}_{max}$ = maximum potential energy
ρ = density of the structural material
W = beam width
h = beam thickness
$\hat{y}(x)$ = resonance mode shape

Example: Folded-Beam Resonator



- Derive an expression for the resonance frequency of the folded-beam structure at left.

Use Rayleigh-Ritz method:

$$KE_{max} = PE_{max}$$

Kinetic Energy:

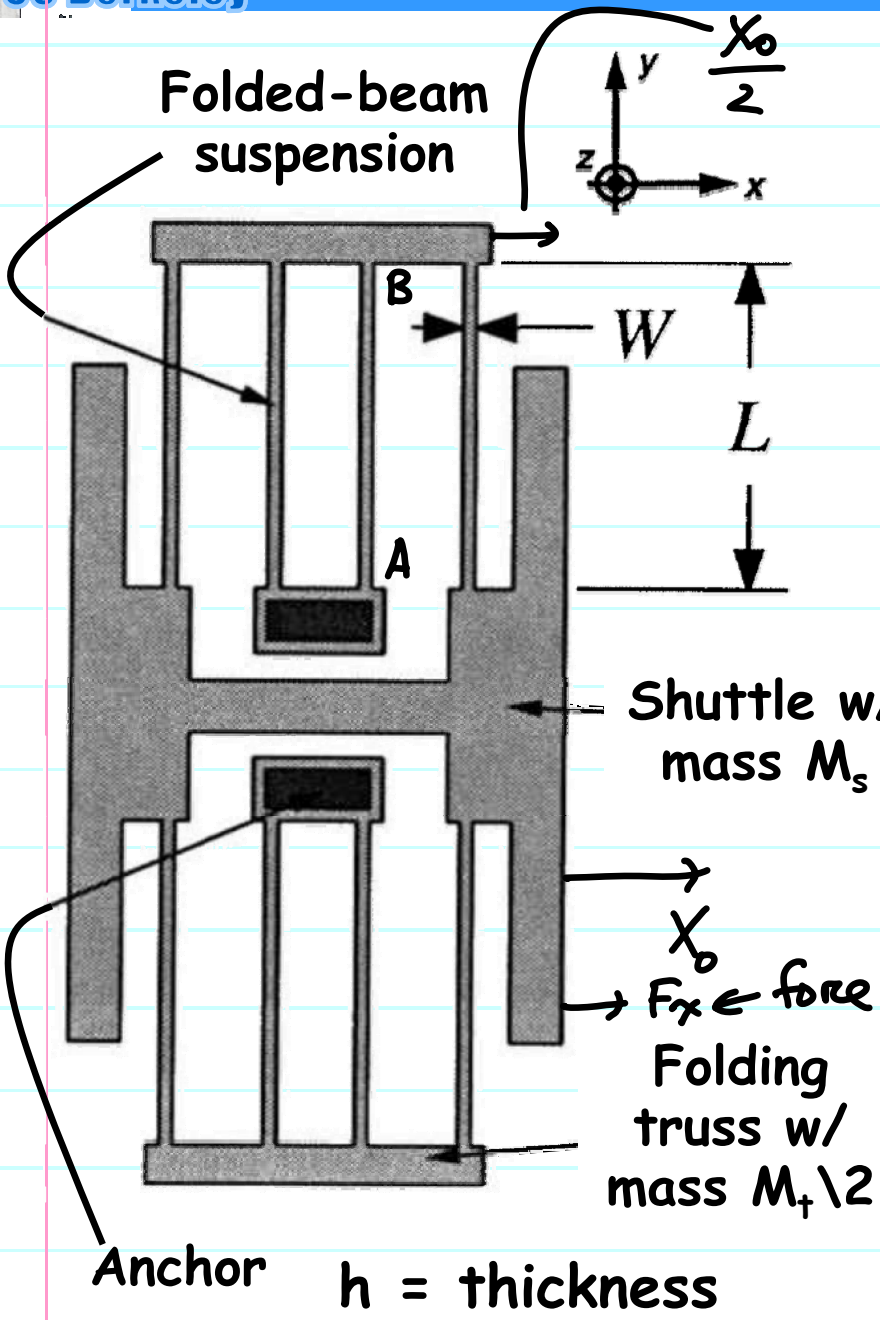
$$KE_{max} = \underbrace{KE_s}_{\text{shuttle}} + \underbrace{KE_t}_{\text{trusses}} + \underbrace{KE_b}_{\text{beams}}$$

$$= \frac{1}{2} v_s^2 M_s + \frac{1}{2} v_t^2 M_t + \underbrace{\frac{1}{2}\int v_b^2 \, dM_b}$$

mass of both trusses

Must integrate since the beam velocity is a function of location y!

Get Kinetic Energies



Velocity of the shuttle: $v_s = \omega_o X_o$

Resonance Freq. → ω_o; Maximum Displacement Amplitude → X_o

ρV

$$\therefore KE_s = \frac{1}{2} v_s^2 M_s = \frac{1}{2}\omega_o^2 X_o^2 M_s$$

Velocity of the truss: $v_t = \frac{1}{2} v_s = \frac{1}{2}\omega_o X_o$

$$\therefore KE_t = \frac{1}{2}\left(\frac{1}{2}\omega_o X_o\right)^2 M_t = \frac{1}{8}\omega_o^2 X_o^2 M_t$$

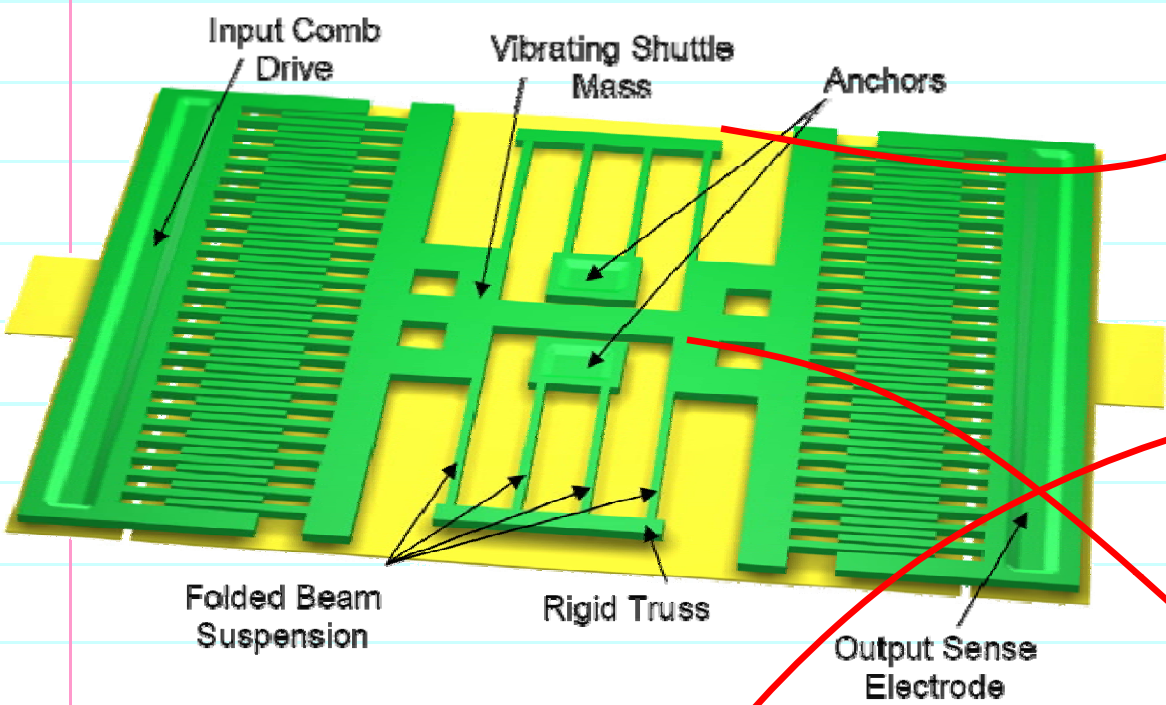
Velocity of the beam segments:

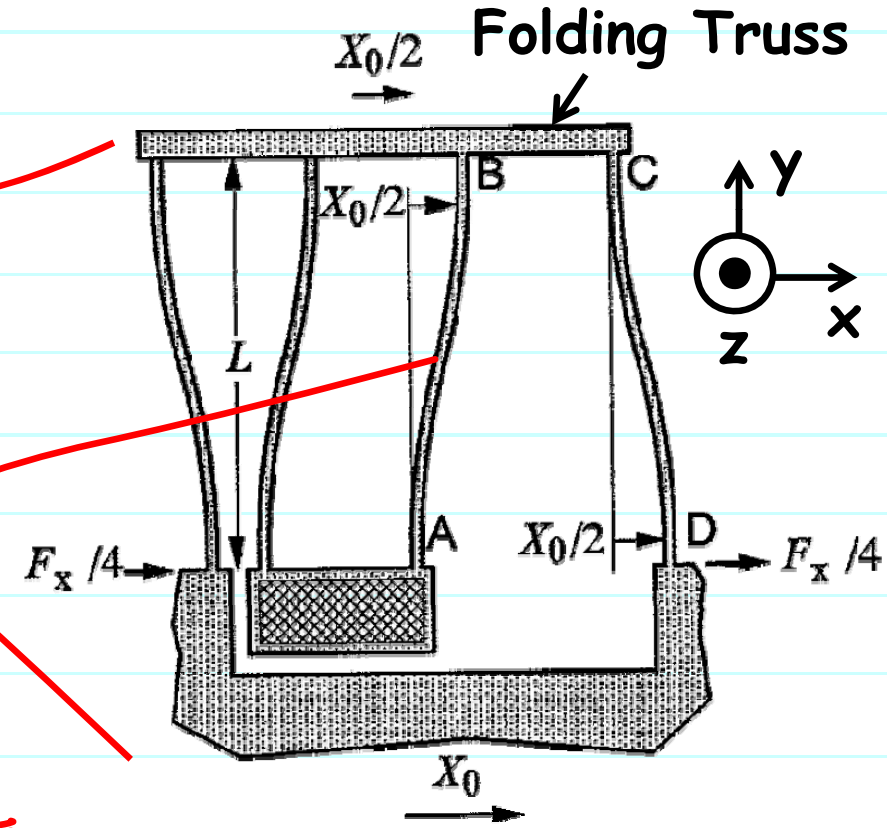
⇒ assume the mode shape is the same as the static displacement shape

⇒ For segment AB

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right) \quad 0 \le y \le L \qquad (1)$$

Folded-Beam Suspension





Comb-Driven Folded Beam Actuator

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right) \qquad 0 \le y \le L$$

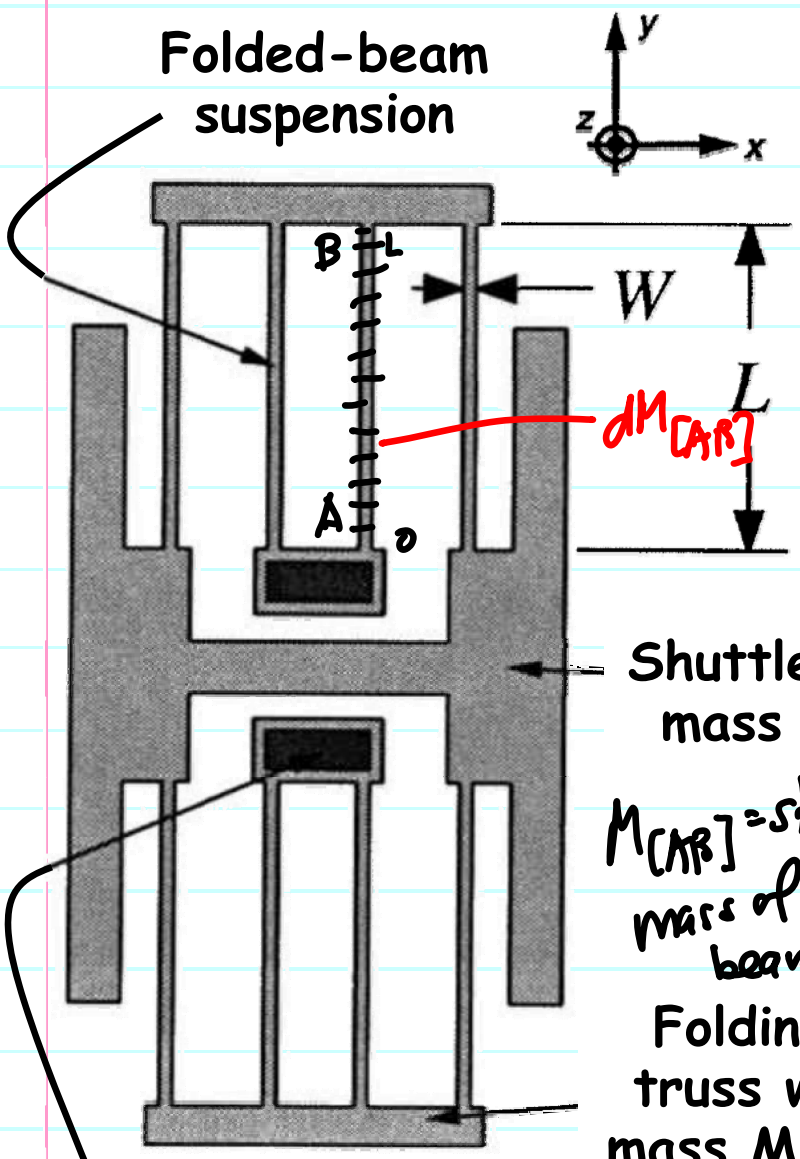
$\frac{F_x/4}{12EI_z}$

Case: $y=0$ $\quad \hat{x}(y=0) = 0$ ✓

Case: $y=L_c$ $\quad \hat{x}(y=L_c) = \frac{F_x}{48EI_z}L^3 \rightarrow k = \frac{48EI_z}{4L^3} = \frac{2k_c}{2}$ ✓

$= \frac{F_x/4}{x}$

Get Kinetic Energies (cont)



h = thickness

At $y = L$: $x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48 E I_z}$

Substituting into (1):

$$\hat{x}(y) = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\Big|_{[AB]} = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_0$$

Plugging into the expression for KE_b:

$$KE_{[AB]} = \frac{1}{2}\int_0^L \frac{X_0^2\omega_0^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$M_{[AB]}$ = static mass of the beam

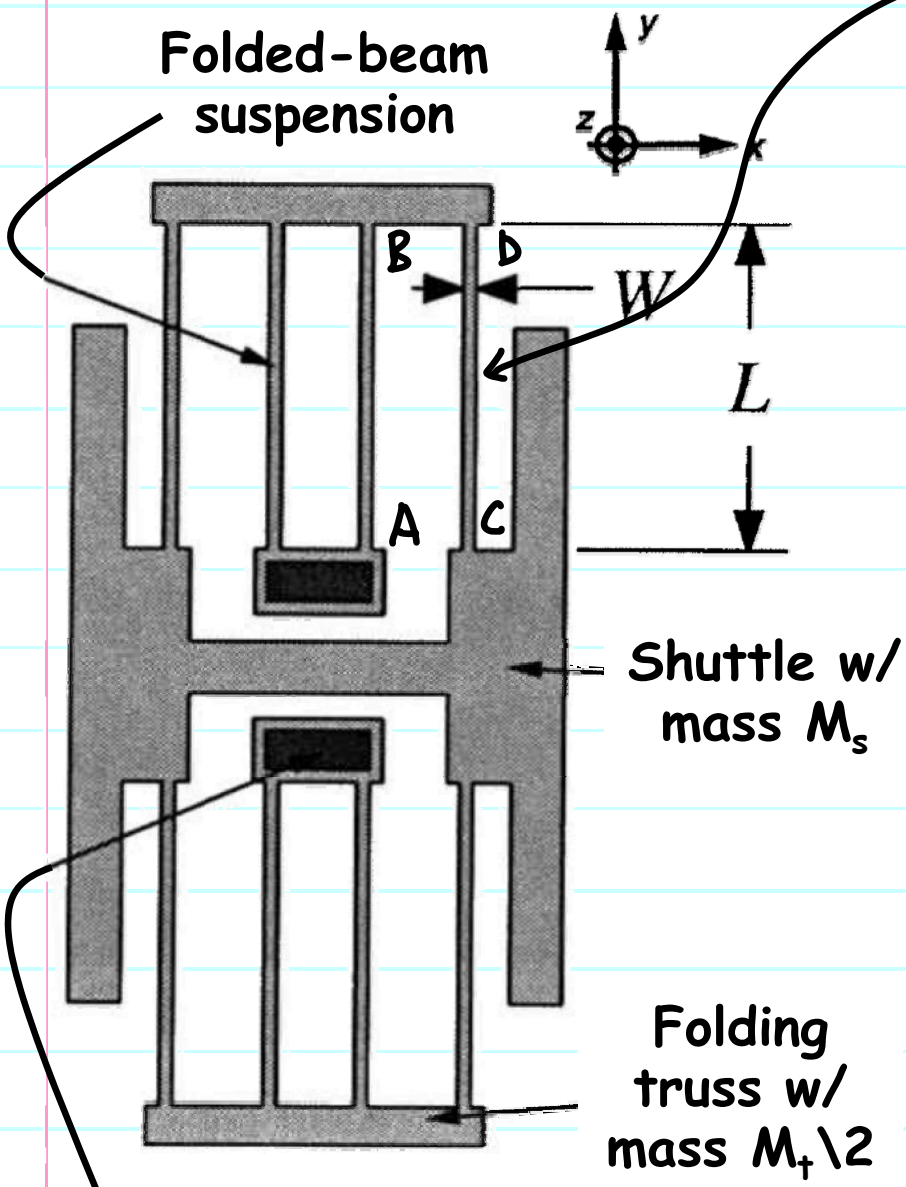
[uniform material] $\Rightarrow dM_{[AB]} = \frac{M_{[AB]}}{L}dy$

ρV = density · volume

$$= \frac{X_0^2\omega_0^2 M_{[AB]}}{8L}\int_0^L\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

$$\Rightarrow KE_{[AB]} = \frac{13}{280}X_0^2\omega_0^2 M_{[AB]}$$

Get Kinetic Energies (cont)



For segment CD:

$$v_b(y)\big|_{[CD]} = X_0\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_0$$

Thus:

$$KE_{[CD]} = \frac{X_0^2\omega_0^2 M_{[CD]}}{2L}\int_0^L \left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]dy$$

$$KE_{[CD]} = \frac{83}{280}X_0^2\omega_0^2 M_{[CD]}$$

Static mass of beam [CD]

Let $M_b \triangleq$ total mass of the 8 beams.

Then: $M_{[AB]} = M_{[CD]} = \frac{1}{8}M_b$

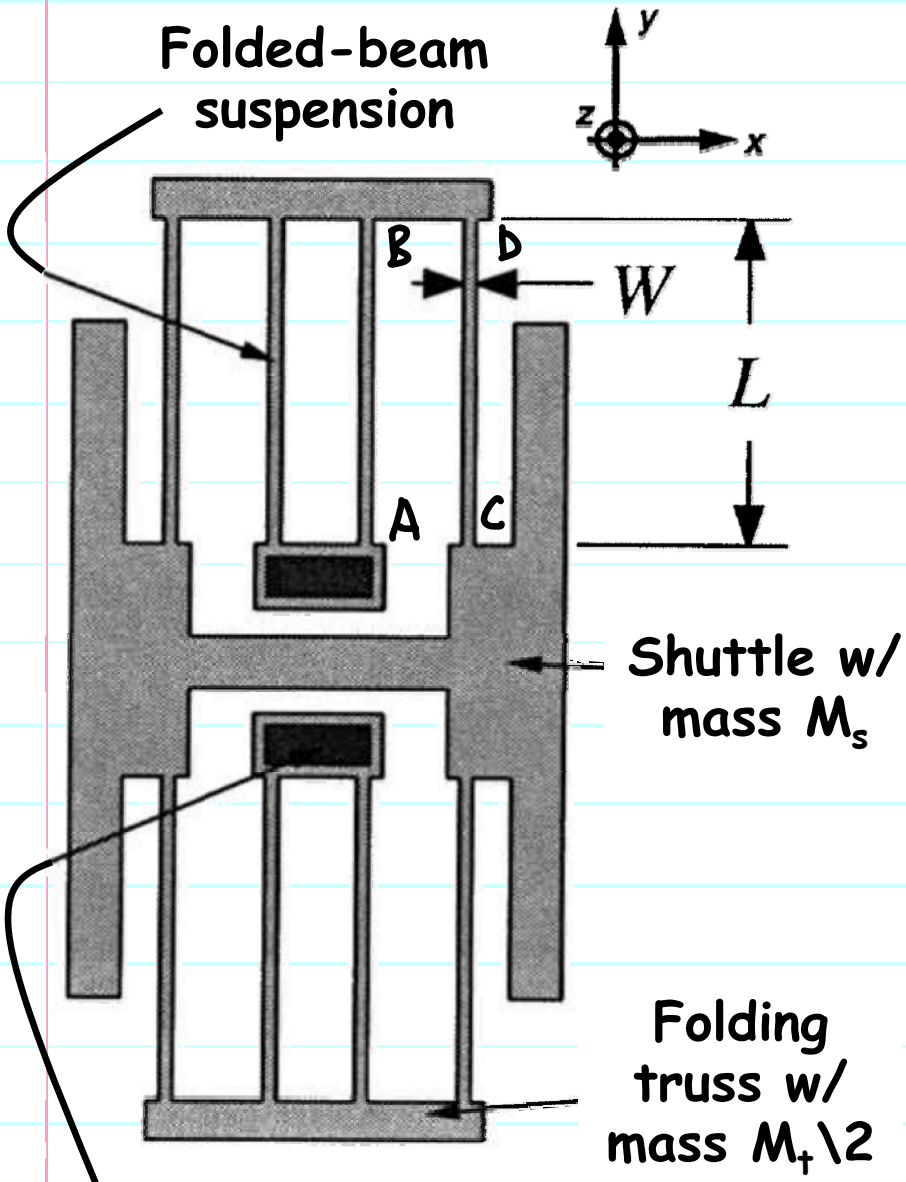
Thus:

$$KE_b = 4KE_{[AB]} + 4K_{[CD]} = \frac{6}{35}X_0^2\omega_0^2 M_b$$

and

$$KE_{max} = X_0^2\omega_0^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right]$$

Get Potential Energy & Frequency



PE_{max} is simply the work done to achieve maximum deflection:

$$PE_{max} = \frac{1}{2} \overset{=k_c}{k_x} X_0^2$$

Thus, using Raleigh-Ritz:

$$KE_{max} = PE_{max}$$

$$\cancel{X_0^2}\omega_0^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right] = \frac{1}{2}\overset{=k_c}{k_x}\cancel{X_0^2}$$

$$\boxed{\omega_0 = \left[\frac{\overset{=k_c}{k_x}}{M_{eq}}\right]^{1/2} \quad \text{where } M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b}$$

(Resonance Frequency of a Folded-Beam Suspended Shuttle)