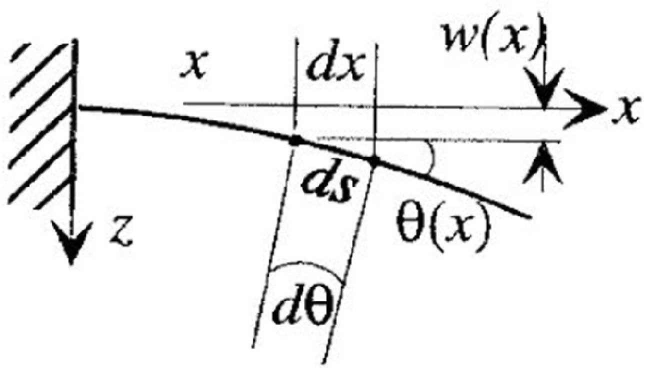
Lecture 13: Beam Bending II

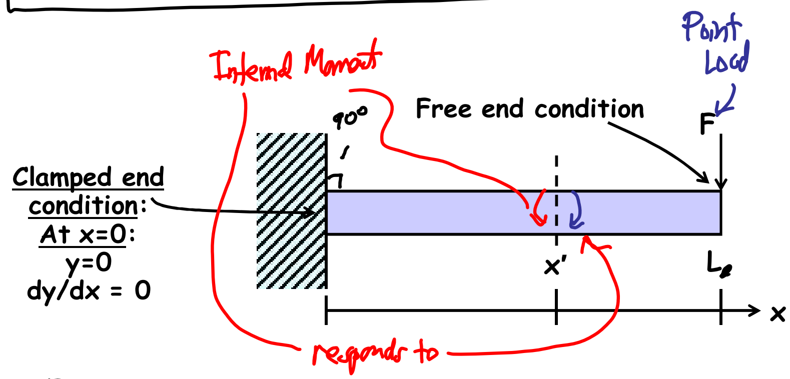
- Announcements:
- HW#3 online, due Thursday, next week, 10 a.m.
  - shorter time span than before
- ---------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- ---------------------------------------
- Last Time:



Inserting (1) into (2):

$$\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$$ ← Diff. Eq. for Small Angle Beam Bending

Cantilever Beam w/ Concentrated Load



Internal Moment @ position x: $M = -F(L-x)$

Thus: $$\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

w/ $\begin{cases} \text{Clamped End B.C.: } w(x=0)=0, \ \frac{dw}{dx}(x=0)=0 \\ \text{Free End B.C.: none} \end{cases}$

Solve to get w:

⇒ use Laplace; or apply a trial solution:

$w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s

$$\boxed{w = \frac{FL}{2EI}x^2\left(1 - \frac{x}{3L}\right)}$$

Deflection at x due to a point load F applied @ $x=L$

Maximum Deflection → occurs @ $x=L$

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \underbrace{\left(\frac{3EI}{L^3}\right)}_{\text{stiffness} \triangleq k_c \text{ @ } x=L} w(x=L)$$

$\rightarrow k_c w(x=L)$

where $\boxed{k_c = \frac{3EI}{L^3}} \triangleq$ stiffness @ $x=L$

In general, stiffness is a function of location

$$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \boxed{k_c = \frac{1}{4}EW\frac{h^3}{L^3}}$$

Ex. $L = 100\mu m$, $W = 2\mu m$, $h = 2\mu m$

polysilicon → $E = 150\,GPa$

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = 0.6\,N/m$$

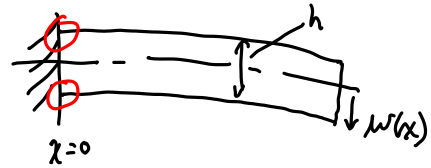
Maximum Stress in a Bent Cantilever

From before, the radius of curvature is given by

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

$\Rightarrow \frac{1}{R}$ is maximized (i.e., R is minimized) when $x=0$:

$$[x=0] \Rightarrow \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$$



Strain is maximized:

1. At top surface @ $x=0$ → tensile
2. At bottom surface @ $x=0$ → compressive

$$\epsilon_{max} = \frac{z}{R} = \frac{h}{2}\frac{1}{R} = \frac{h}{2}\frac{FL}{EI} = \epsilon_{max}$$

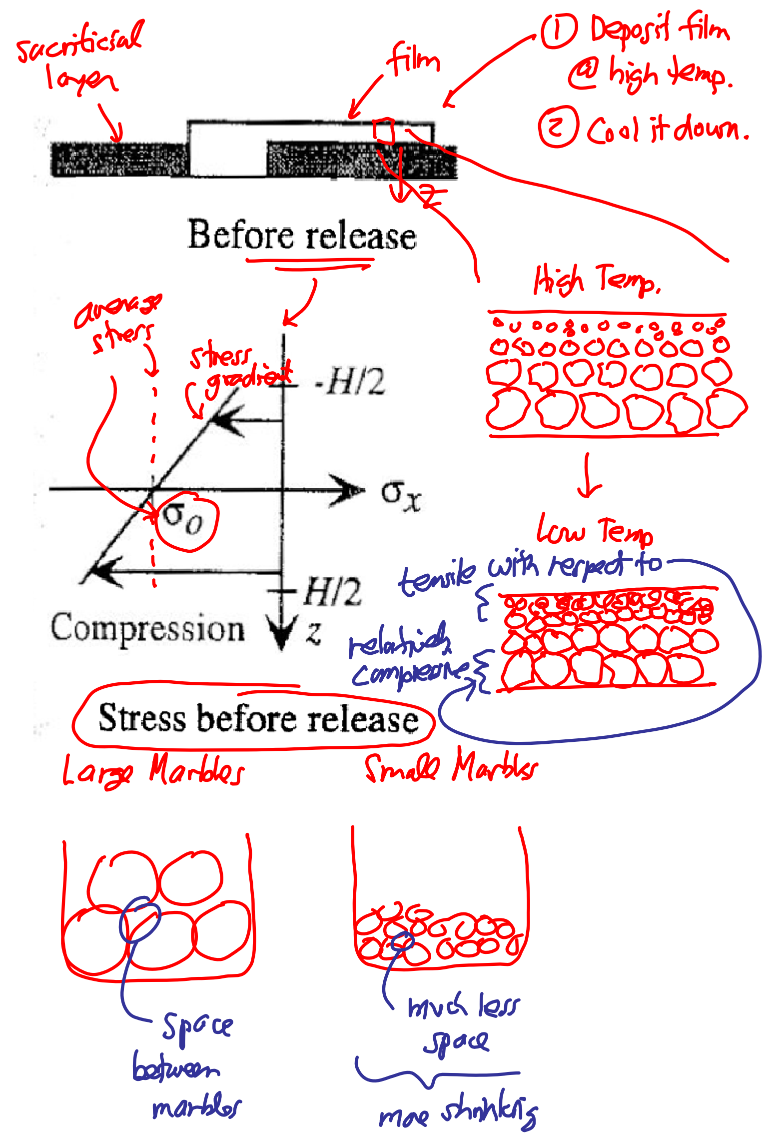
$$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \epsilon_{max} = \frac{\cancel{h}}{2}\frac{FL}{E}\left(\frac{12}{Wh^{\cancel{3}}}\right) = \frac{6L}{EWh^2}F$$

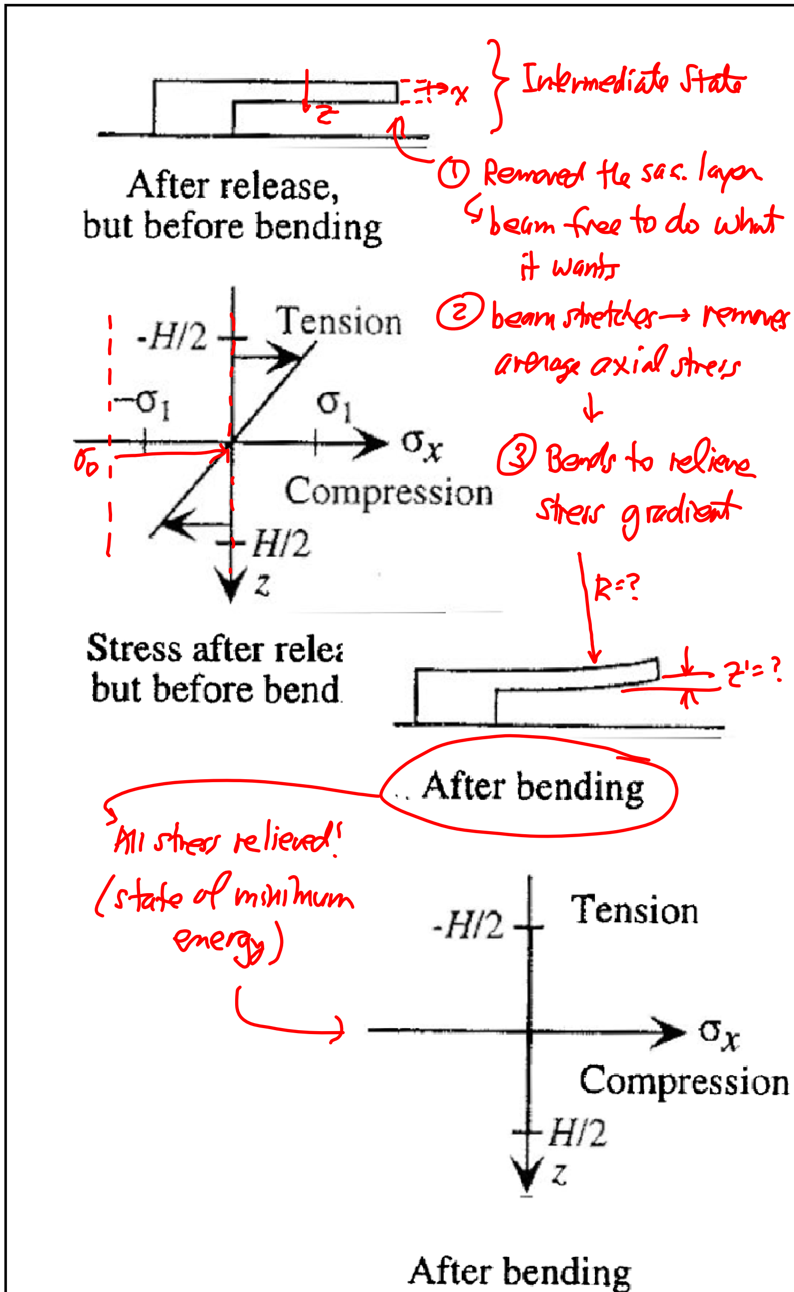
$$\sigma_{max} = \epsilon_{max}E = \frac{6L}{Wh^2}F$$

(maximum stress in a bent cantilever subjected to a force F at its tip)

Stress Gradient in a Cantilever

⇒ study beam bending due a stress gradient





Bending Due to a Stress Gradient

Find the radius of curvature.

Prior to release, axial stress: $\sigma = \sigma_0 - \frac{\sigma_1}{(H/2)} z$

The internal moment:

$$M_x = \int_{-\frac{H}{2}}^{\frac{H}{2}} [(W dz)\sigma] z = \int_{-\frac{H}{2}}^{\frac{H}{2}} W\left(z\sigma_0 - \frac{\sigma_1 z^2}{(H/2)}\right) dz$$

$$= W\left(\frac{1}{2}\sigma_0 z^2 - \frac{2\sigma_1 z^3}{3H}\right)\Bigg|_{-\frac{H}{2}}^{\frac{H}{2}}$$

$$= W\left(\cancel{\frac{1}{2}\sigma_0\frac{H^2}{4}} - \frac{2}{3}\sigma_1\frac{H^2}{8} - \cancel{\frac{1}{2}\sigma_0\frac{H^2}{4}} - \frac{2}{3}\frac{\sigma_1 H^2}{8}\right)$$

average stress cancels out

$$\boxed{M_x = -\frac{1}{6}\sigma_1 W H^2}$$

Thus, the radius of curvature:

$$\frac{1}{R} = -\frac{M_x}{E'I} \rightarrow R = \frac{E'I}{M_x} = \frac{1}{2}\frac{E'H}{\sigma_1}$$

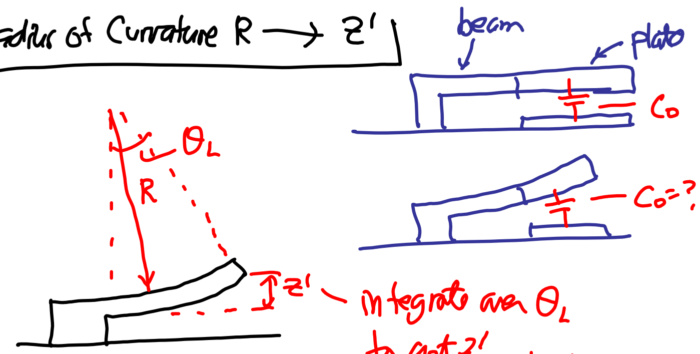
Biaxial Modulus

$\left[I = \frac{1}{12} W h^3\right]$

$$\boxed{R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1}}$$

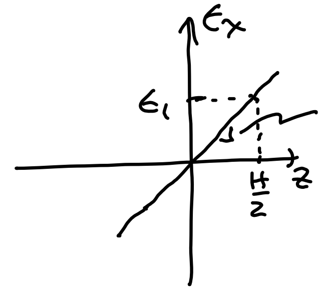
[Radius of Curvature for a Cantilever w/ a Stress Gradient]

Radius of Curvature $R \rightarrow z'$



integrate over θ_L to get z' → Do this in your HW

Definition. Strain Gradient



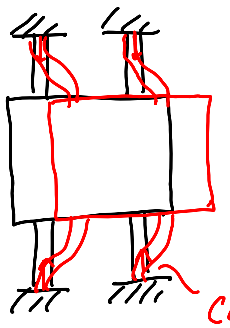
slope $\triangleq$ Strain Gradient $= \Gamma$

$$\Gamma = \frac{\epsilon_1}{(H/2)}$$

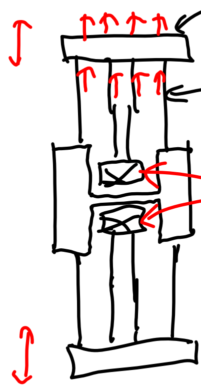
$$R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1} = \frac{H}{2}\frac{E'}{\sigma_1} = \frac{(H/2)}{\epsilon_1} = \frac{1}{\Gamma} \rightarrow \boxed{\Gamma = \frac{1}{R}}$$

Clamped-Clamped Supports

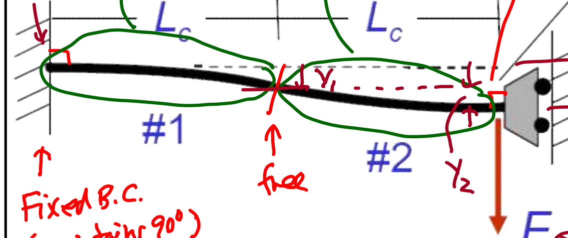
⇒ not good for stress



Smarter: fold the beams



Series Combination of Springs



Series: $y_{tot} = y_1 + y_2$ (from anchor pt. to forcing pt.)

or $y_{tot} > y_1$ or $y_{tot} > y_2$