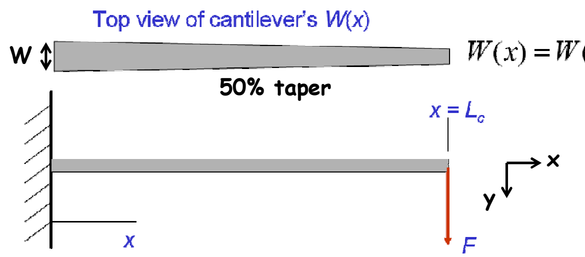
More General Geometries

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam
- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



$$W(x) = W(1 - \frac{x}{2L_c})$$

Solution: Use Principle of Virtual Work

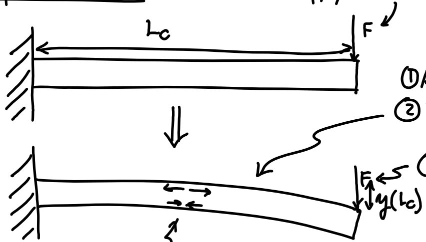
- In an energy-conserving system (i.e., elastic materials), the energy stored in a body due to the quasi-static (i.e., slow) action of surface and body forces is equal to the work done by these forces …
- Implication: if we can formulate stored energy as a function of the deformation of a mechanical object, then we can determine how an object responds to a force by determining the shape the object must take in order to minimize the difference U between the stored energy and the work done by the forces:

U = Stored Energy - Work Done

- Key idea: we don't have to reach U = 0 to produce a very useful, approximate *analytical* result for load-deflection

More Visual Description …

Same problem as before: Take a beam & apply a force:



① Apply force.
② Beam responds by bending.
③ This force has done work:
$W = F \cdot y(L_c)$
④ Strain generated → This means the beam has received an influx of stored energy
magnitude determined by its deformed shape
⑤ Then:
U = Stored Energy – Work Done → 0
When we choose the right shape! (This is how we get the beam's response to F!)

Fundamentals: Energy Density

- Strain energy density: [J/m³]
  - To find work done in straining material

$W(Q) = \int_0^Q \frac{Q}{C} dQ$ → charging a capacitor from $0 \to Q$ takes this much work
→ stored energy on a capacitor

value of strain @ position (x,y,z)

$$w = \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x \quad \text{x-axis normal stress term}$$

$\sigma_x(\varepsilon_x)$ → relates stress to strain @ position (x,y,z)

$$[\sigma_x = E\epsilon_x] \Rightarrow \quad w = \int_0^{\varepsilon_x} E\varepsilon_x d\varepsilon_x = \frac{1}{2}E\varepsilon_x^{\;2}$$

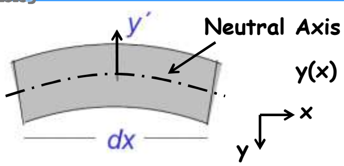
This is a definition, so really can just say it's a definition.

$W(q_1) = \int_0^{q_1} e(q)dq$ q = displacement, e = effort } Generic Definition of Work

- Total strain energy [J]:
  - Integrate over all strains (normal and shear)

$$W = \iiint\left(\frac{1}{2}E\left(\varepsilon_x^{\;2}+\varepsilon_y^{\;2}+\varepsilon_z^{\;2}\right)+\frac{1}{2}G\left(\gamma_{xy}^{\;2}+\gamma_{xz}^{\;2}+\gamma_{yz}^{\;2}\right)\right)dV$$

Bending Energy Density



y(x) = transverse displacement of neutral axis

- First, find the bending energy dW_{bend} in an infinitesimal length dx:

W = width

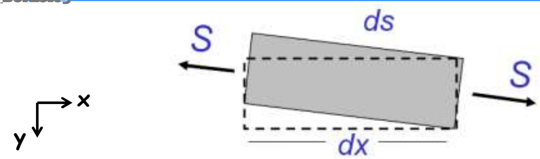
$$dW_{bend} = W\,dx \int_{-h/2}^{h/2} \frac{1}{2} E \epsilon_x^2(y')\,dy'$$

$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \Rightarrow \epsilon_x(y') = y' \frac{d^2y}{dx^2}$$

$$dW_{bend} = W\,dx \int_{-h/2}^{h/2} \frac{1}{2} E \left[y' \frac{d^2y}{dx^2}\right]^2 dy' = \frac{1}{2} E \underbrace{\left(\frac{Wh^3}{12}\right)}_{I_z} \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{W_{bend} = \frac{1}{2} E I_z \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}$$

Energy Due to Axial Load



- Strain due to axial load S contributes an energy $dW_{stretch}$ in length dx, since lengthening of the different element dx (to ds) results in a strain ε_x

Binomial Theorem

$$ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} \approx dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore \epsilon_x = \frac{ds - dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

Axial Strain Energy

$$\left[dW_{axial} = S\epsilon_x\,dx = \frac{1}{2} S \left(\frac{dy}{dx}\right)^2 dx\right] \Rightarrow \boxed{W_{axial} = \frac{1}{2} S \int_0^L \left(\frac{dy}{dx}\right)^2 dx}$$

Shear Strain Energy

$$\boxed{W_{shear} = \frac{3(E I_z)^2}{4 G W h} \int_0^L \left(\frac{d^3 y}{dx^3}\right)^2 dx}$$

Shear Modulus

- See W.C. Albert, "Vibrating Quartz Crystal Beam Accelerometer," Proc. ISA Int. Instrumentation Symp., May 1982, pp. 33-44

Applying the Principle of Virtual Work

- Basic Procedure:
  - Guess the form of the beam deflection under the applied loads → guess $y(x,F)$
  - Vary the parameters in the beam deflection function in order to minimize:

Assumes point load

$$U = \underbrace{\sum_j W_j}_{\text{Sum strain energies}} - \sum_i F_i u_i$$

Displacement at point load

  - Find minima by simply setting derivatives to zero
- See Senturia, pg. 244, for a general expression with distrubuted surface loads and body forces

Example: Tapered Cantilever Beam

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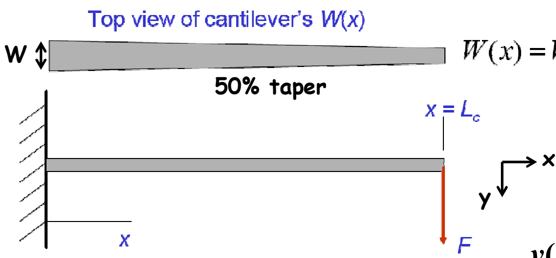


$$W(x) = W(1 - \frac{x}{2L_c})$$

Adjustable parameters: minimize U

$$y(x) = c_2 x^2 + c_3 x^3$$

Find

- Start by guessing the solution
  - It should satisfy the boundary conditions
  - The strain energy integrals shouldn't be too tedious
    - This might not matter much these days, though, since one could just use matlab or mathematica

Strain Energy And Work By F

$$U = W_{bend} - F \cdot y(L_c)$$

$$W_{bend} = \frac{1}{2} E \int_0^{L_c} I_z(x) \left(\frac{d^2 y}{dx^2} \right)^2 dx \quad \text{(Bending Energy)}$$

$$I_z(x) = \frac{W(x) h^3}{12} \qquad \frac{d^2 y}{dx^2} = 2c_2 + 6c_3 x$$

(Using our guess)

$$W(x) = W(1 - \frac{x}{2L_c})$$

$$= \frac{1}{24} E W h^3 \int_0^{L_c} (1 - \frac{x}{2L_c})(2c_2 + 6c_3 x)^2 dx - \underbrace{F \overbrace{(c_2 L_c^2 + c_3 L_c^3)}^{\text{Tip Deflection}}}_{\text{work}}$$

Find c_2 and c_3 That Minimize U

- Minimize U → basically, find the c_2 and c_3 that brings U closest to zero (which is what it would be if we had guessed correctly)
- The c_2 and c_3 that minimize U are the ones for which the partial derivatives of U with respective to them are zero:

$$\frac{\partial U}{\partial c_2} = 0 \qquad \frac{\partial U}{\partial c_3} = 0$$

- Proceed:
  - First, evaluate the integral to get an expression for U:

$$U = EWh^3 \left\{ \frac{5c_3^2}{16} L_c^3 + \frac{c_2 c_3}{3} L_c^2 + \frac{c_2^2}{8} L_c \right\} - F\left(c_2 L_c^2 + c_3 L_c^3\right)$$

Minimize U (cont)

- Evaluate the derivatives and set to zero:

$$\frac{\partial U}{\partial c_2} = 0 = \left(\frac{EWh^3}{3} c_3 - F \right) L_c^2 + \left(\frac{EWh^3}{4} c_2 \right) L_c$$

$$\frac{\partial U}{\partial c_3} = 0 = \left(\frac{5}{8} EWh^3 c_3 - F \right) L_c^3 + \left(\frac{EWh^3}{3} c_2 \right) L_c^2$$

- Solve the simultaneous equations to get c_2 and c_3:

$$\boxed{c_2 = \left(\frac{84}{13} \right) \frac{F L_c}{EWh^3} \qquad c_3 = -\left(\frac{24}{13} \right) \frac{F}{EWh^3}}$$

The Virtual Work-Derived Solution

- And the solution:

$$y(x) = \left(\frac{24F}{13EWh^3}\right)\left(\left(\frac{7}{2}\right)L_c - x\right)x^2$$

- Solve for tip deflection and obtain the spring constant:

$$y(L_c) = \left(\frac{24F}{13EWh^3}\right)\left(\frac{5}{2}\right)L_c^{\ 3} \qquad k_c = F / y(L_c) = \left(\frac{13EWh^3}{60L_c^{\ 3}}\right)$$

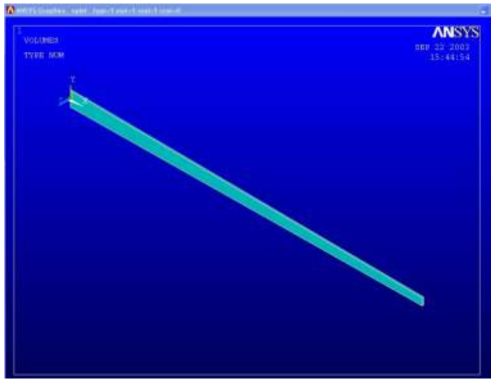
- Compare with previous solution for constant-width cantilever beam (using Euler theory):

$$y(L_c) = \left(\frac{4F}{EWh^3}\right)L_c^{\ 3} \longrightarrow$$ 13% smaller than tapered-width case

Comparison With Finite Element Simulation

- Below: ANSYS finite element model with

L = 500 μm W_{base} = 20 μm E = 170 GPa
h = 2 μm W_{tip} = 10 μm



- Result: (from static analysis)
  - k = 0.471 μN/m
- This matches the result from energy minimization to 3 significant figures

Need a Better Approximation?

- Add more terms to the polynomial
- Add other strain energy terms:
  - Shear: more significant as the beam gets shorter
  - Axial: more significant as deflections become larger
- Both of the above remedies make the math more complex, so encourage the use of math software, such as Mathematica, Matlab, or Maple
- Finite element analysis is really just energy minimization
- If this is the case, then why ever use energy minimization analytically (i.e., by hand)?
  - Analytical expressions, even approximate ones, give insight into parameter dependencies that FEA cannot
  - Can compare the importance of different terms
  - Should use in tandem with FEA for design