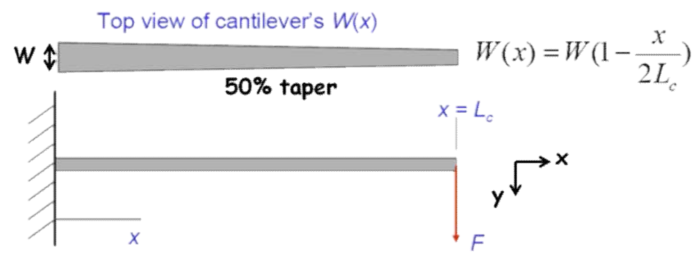
Lecture 16: Energy Methods

- Announcements:
- HW#4 online, due Thursday, next week, 10 a.m.
- Module 9 on "Energy Methods" online
- Midterm Exam less than 2 weeks away, Tuesday, March 21, 3:30-5 p.m., 521 Cory (right here)
- --
- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
- --
- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
- --
- Last Time:
- Started into Module 9 on "Energy Methods"
- Continue with this

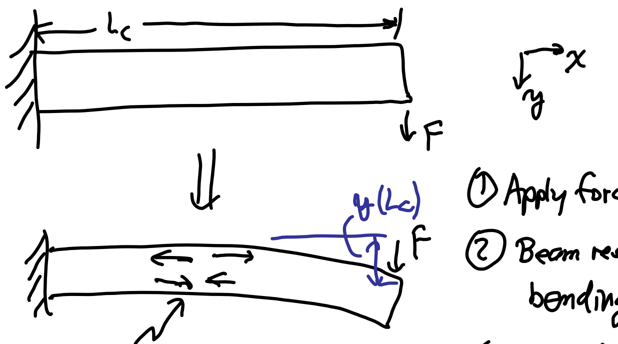
More General Geometries

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam
- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



over

Same problem before: Take a beam, apply a force.



① Apply force F.

② Beam responds by bending.

③ This force has done work:

$$\mathcal{W} = F \cdot y(L_c)$$

④ Strain generated
↳ so the beam has received an influx of energy
↳ magnitude of energy determines the shape taken by beam

⑤ Then

$$U = \underbrace{\text{Stored Energy}}_{\text{function of its shape}} - \text{Work Done} \rightarrow 0$$

when this = 0, we have the right shape

Get transfer fcn

$y(x) = f(x)$

??? How to determine this?

Fundamentals: Energy Density

General Definition of Work:

$$\mathcal{W}(q_1) = \int_0^{q_1} e(q)\,dq$$

q = displacement
e = effort

↳ for EE: $\mathcal{W}(Q) = \int_0^Q \frac{Q}{C}\,dQ$

Strain Energy Density

$$w = \int_0^{\epsilon_x} \sigma_x\, d\epsilon_x$$

ϵ_x ← value of strain @ position (x,y,z)

$\sigma_x(\epsilon_x)$ ← relates stress to strain @ position (x,y,z)

$[\sigma_x = E\epsilon_x]$

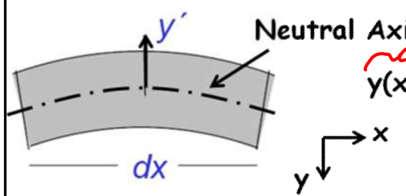
$$w = \int_0^{\epsilon_x} E\epsilon_x\, d\epsilon_x = \frac{1}{2}E\epsilon_x^2$$

Total Strain Energy: [J]

$$\mathcal{W} = \iiint \left(\frac{1}{2}E\left(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2\right) + \frac{1}{2}G\left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right)\right) dV$$

volume ↓ dV

Bending Energy Density



same as z before

y(x) = transverse displacement of neutral axis

First, find the bending energy dW_{bend} in an infinitesimal length dx:

$$dW_{bend} = W\,dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} E \epsilon_x^2(y')\,dy'$$

$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \Rightarrow \epsilon_x(y') = y'\frac{d^2y}{dx^2}$$

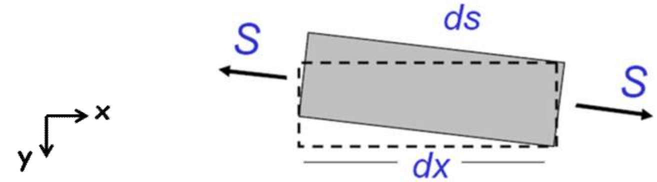
$$dW_{bend} = W\,dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} E \left[y'\frac{d^2y}{dx^2}\right]^2 dy'$$

$$= \frac{1}{2} E \underbrace{\left(\frac{Wh^3}{12}\right)}_{I_z} \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{W_{bend} = \frac{1}{2} E I_z \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}$$

→ Strain Energy in a Bent Beam

Energy Due to Axial Load



⇒ energy related to lengthening:

$$ds = \left[(dx)^2 + (dy)^2\right]^{\frac{1}{2}} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$$

binomial theorem

$$\approx dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore \epsilon_x = \frac{ds - dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

$$\left[dW_{axial} = S\epsilon_x dx = \frac{1}{2} S \left(\frac{dy}{dx}\right)^2 dx\right]$$

$$\boxed{W_{axial} = \frac{1}{2} S \int_0^L \left(\frac{dy}{dx}\right)^2 dx}$$

Strain Energy Due to Axial Load

⇒ Look @ shear strain energy in your module.

Estimating Resonance Frequency

$\mapsto x(t) = X_o \cos\omega t$

Potential Energy:

$$W(t) = \frac{1}{2} k x^2(t) = \frac{1}{2} k X_o^2 \cos^2 \omega t$$

Kinetic Energy:

$$K(t) = \frac{1}{2} m [\dot{x}(t)]^2 = \frac{1}{2} m X_o^2 \omega^2 \sin^2 \omega t$$

$\dot{x} = \frac{dx}{dt} = \text{velocity}$

Remarks:

① Energy must be conserved.

② Total Energy = Potential Energy + Kinetic Energy at all times & locations on the structure

$$W_{max} = \frac{1}{2} k X_o^2 = K_{max} = \frac{1}{2} m \omega^2 X_o^2$$

maximum potential energy

peak displacement ($K=0$)

maximum kinetic energy

maximum velocity ($W=0$)

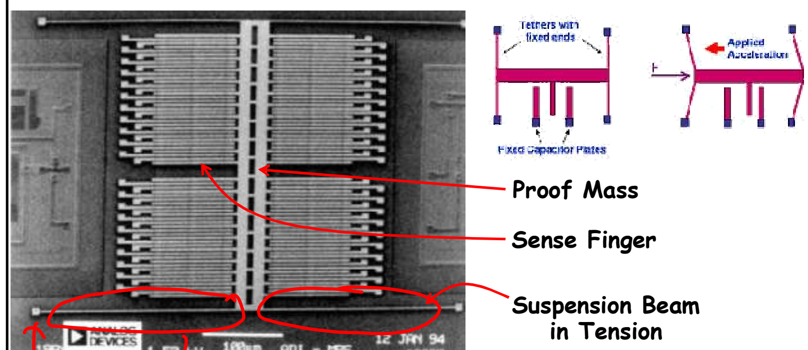
*

* $\boxed{\omega_o = \sqrt{\frac{k}{m}}}$ ⇒ good for problems where mass & stiffness can be separated, i.e., are distinct

radian resonance frequency

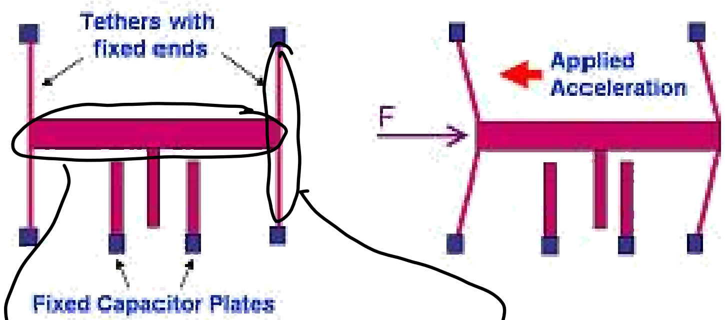
Example: ADXL-50

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



In fabrication, purposely introduce a tensile stress in the beams!

a rather large one!



mass of structure >> mass of springs

∴ ignore the mass of springs

stiffness of springs << stiffness of the structure

∴ ignore the stiffness of the structure

For ADXL-50: 60% of the mass comes from the sense fingers → M = 162 ng

Suspension: Four tensioned beams