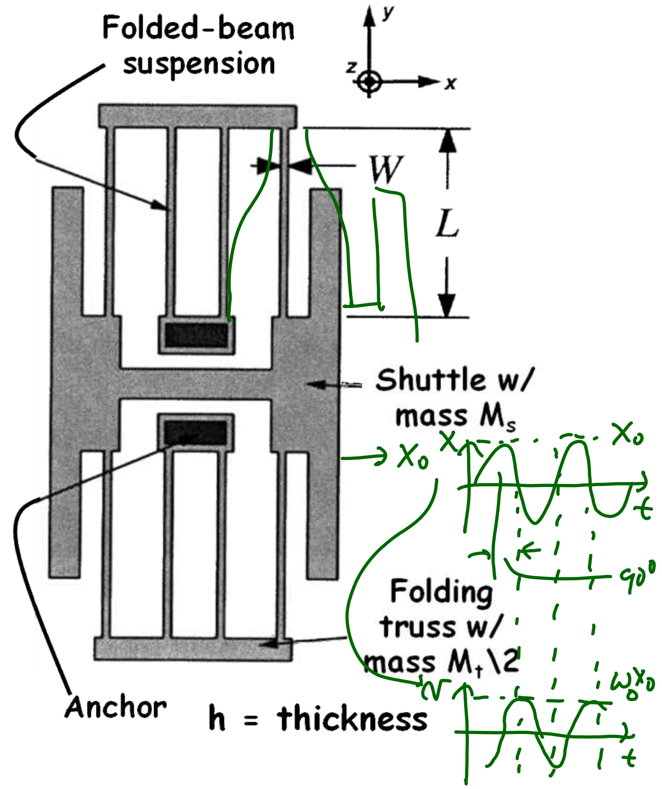
## Lecture 18: Equivalent Circuits I

- Announcements:
- Module 11 on Equivalent Circuits online
- HW#4 online, now due this coming Monday, 10 a.m., after which you will get solutions
- Midterm Exam next week, Thursday, March 23, 3:30-5 p.m., 521 Cory (right here)
- Passed out solutions to one more old midterm
- Last midterm solutions will be available on Monday in the box outside my office after 10 a.m.
- -----------------------------------
- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - ✧ Estimating Resonance Frequency
  - ✧ Lumped Mass-Spring Approximation
  - ✧ ADXL-50 Resonance Frequency
  - ✧ Distributed Mass & Stiffness
  - ✧ Folded-Beam Resonator
  - ✧ Resonance Frequency Via Differential Equations
- -----------------------------------
- Reading: Senturia, Chpt. 5
- Lecture Topics:
  - ✧ Lumped Mechanical Equivalent Circuits
  - ✧ Electromechanical Analogies
- -----------------------------------
- Last Time:
- Determined resonance frequency for a folded-beam suspended device
- Now, go through Module 10, slides 21-31

## Resonance Freq. of a Folded-Beam Resonator

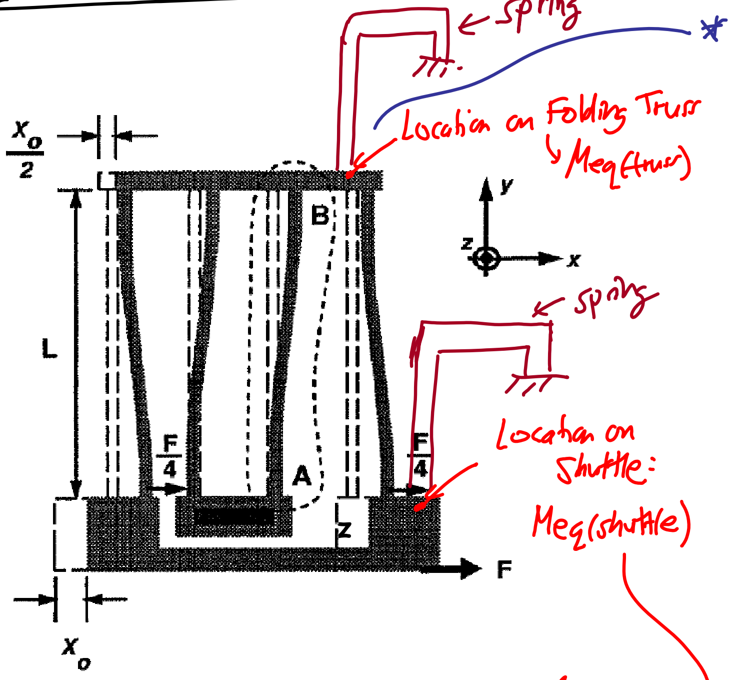


$$\omega_0 = \left[\frac{k_c}{M_{eq}}\right]^{1/2}$$

where $M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b$

(Resonance Freq. of a Folded-Beam Suspended Shuttle)

## Equivalent Dynamic Mass



Equivalent Dynamic Mass:

$$\text{Equiv. Mass} = M_{eq,x} = \frac{KE_{max}}{\frac{1}{2}V_x^2} = \frac{\frac{1}{2}\rho A\int_0^{l}[V(x)]^2dx}{\frac{1}{2}V_x^2}$$

↑ velocity @ location x

$$M_{eq}(shuttle) = \frac{KE_{max}}{\frac{1}{2}V_{shuttle}^2} = \frac{\cancel{\omega_0^2 x_0^2\left(\frac{1}{2}\right)}\left(M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right)}{\cancel{\frac{1}{2}\omega_0^2 x_0^2}}$$

$$\boxed{M_{eq}(shuttle) = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b}$$

total static masses

*

$$M_{eq}(truss) = \frac{\cancel{\omega_0^2 x_0^2\left(\frac{1}{2}\right)}\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}{\cancel{\frac{1}{2}}\left(\frac{1}{4}\right)\cancel{\omega_0^2 x_0^2}}$$

$$\boxed{M_{eq}(truss) = 4\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}$$

↑ Equivalent Dynamic Mass

## Equiv. Dynamic Stiffness

$$\omega_0 = \sqrt{\frac{K_{eq}(x)}{M_{eq}(x)}} \rightarrow \boxed{K_{eq}(x) = \omega_0^2 M_{eq}(x)}$$

⇒ large equiv. mass & large equiv. stiffness go hand-in-hand

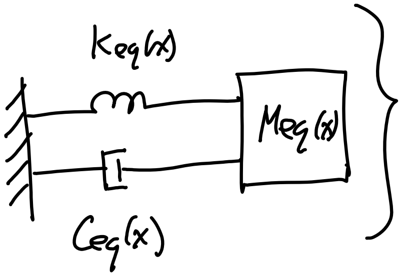
## Equiv. Dynamic Damping

$$Q = \frac{\omega_0 M_{eq}(x)}{C_{eq}(x)} \rightarrow \boxed{C_{eq}(x) = \frac{\omega_0 M_{eq}(x)}{Q} = \frac{\sqrt{k_{eq}(x)M_{eq}(x)}}{Q}}$$

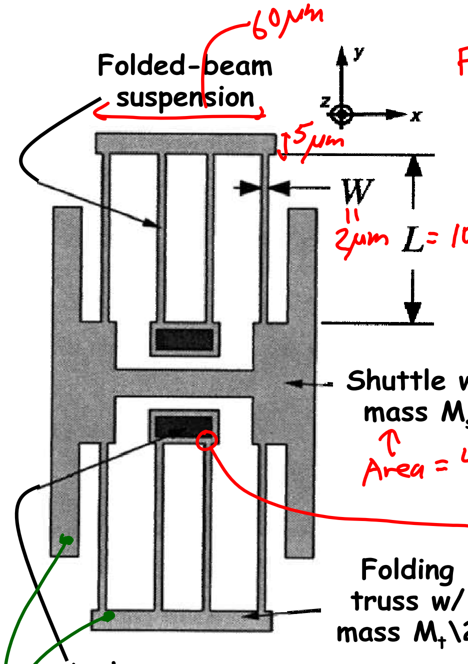
~L

↑ damping

↘ R



specified @ a single location x



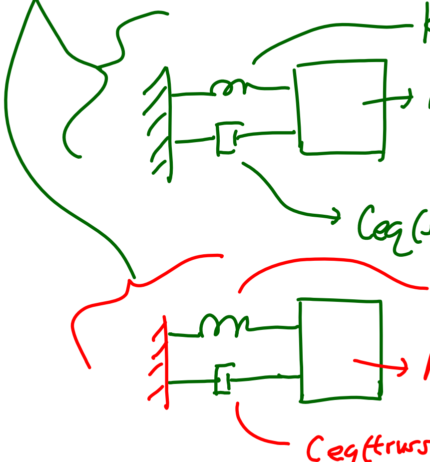
For polysilicon, Q = 100,000

$M_{eq} = \infty$

$K_{eq} = \infty$

$C_{eq} = \infty$

h = thickness



$K_{eq}(\text{shuttle}) = 4.8\ \text{N/m}$

$M_{eq}(\text{shuttle}) = 2.16\times10^{-11}\ \text{kg}$

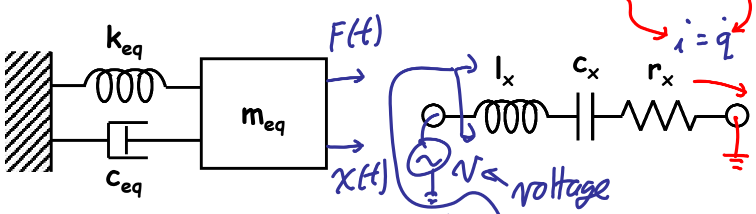
$C_{eq}(\text{shuttle}) = 1.02\times10^{-10}\ \text{kg/s}$

$K_{eq}(\text{truss}) = 19.2\ \text{N/m}$

$M_{eq}(\text{truss}) = 8.64\times10^{-11}\ \text{kg}$

$C_{eq}(\text{truss}) = 4.08\times10^{-10}\ \text{kg/s}$

## Electromechanical Analogies



$F(t) = F\cos(\omega t) \rightarrow x(t) = X\cos(\omega t)$
(off resonance)

Equation of Motion:

$$m_{eq}\ddot{x} + C_{eq}\dot{x} + k_{eq}x = F(t)$$

⇒ using phasor concepts:

$$\boxed{F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + C_{eq}\dot{x}}$$

Impedance looking in

$$\frac{v}{i} = j\omega l_x + \frac{1}{j\omega C_x} + r_x$$

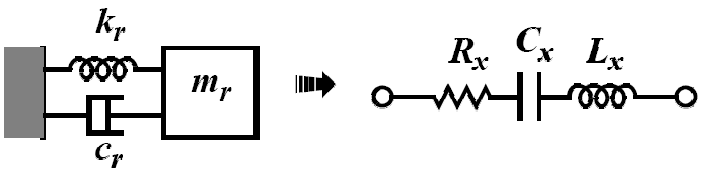
$$\boxed{v = j\omega l_x i + \frac{(1/C_x)}{j\omega} i + r_x i}$$

compare

⇒ by analogy:

$$\boxed{\begin{array}{lll} F \rightarrow v & m_{eq} \rightarrow l_x & \\ & & C_{eq} \rightarrow r_x \\ \dot{x} \rightarrow i & k_{eq} \rightarrow \frac{1}{C_x} & \end{array}}$$

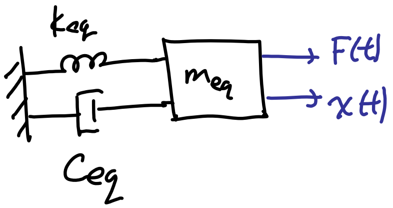
[Parameter Relationships in the Current Analogy]



- Mechanical-to-electrical correspondence in the current analogy:

| Mechanical Variable | Electrical Variable |
|---|---|
| Damping, $c$ | Resistance, $R$ |
| Stiffness$^{-1}$, $k^{-1}$ | Capacitance, $C$ |
| Mass, $m$ | Inductance, $L$ |
| Force, $f$ | Voltage, $V$ |
| Velocity, $v$ | Current, $I$ |

## Lowpass Biquad Transfer Function



$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + C_{eq}\dot{x}$$

⇒ convert to full phasor form:

$$F = (j\omega)(j\omega X)m_{eq} + \frac{k_{eq}}{\cancel{j\omega}}(\cancel{j\omega}X) + C_{eq}(j\omega X)$$

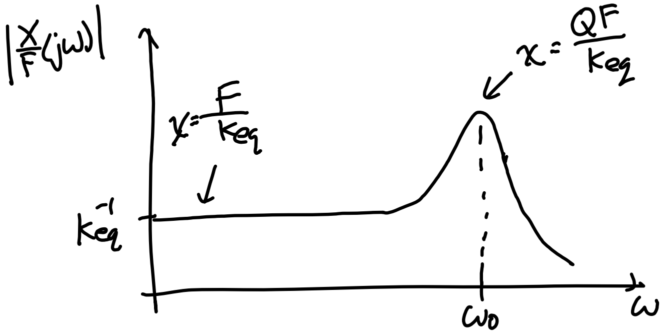
$$\frac{X}{F}(j\omega) = \frac{1}{k_{eq}}\left[-\omega^2\frac{m_{eq}}{k_{eq}} + 1 + j\frac{C_{eq}\omega}{k_{eq}}\right]^{-1} \quad *$$

$$\left[\frac{k_{eq}}{m_{eq}} = \omega_0^2,\ Q = \frac{m_{eq}\omega_0}{C_{eq}} = \frac{k_{eq}}{\omega_0 C_{eq}} \rightarrow \frac{k_{eq}}{C_{eq}} = Q\omega_0\right]$$

$$* \quad \frac{X}{F}(j\omega) = \frac{1}{k_{eq}}\left[-\left(\frac{\omega}{\omega_0}\right)^2 + 1 + j\frac{\omega}{Q\omega_0}\right]^{-1}$$

$$\boxed{\frac{X}{F}(j\omega) = \frac{k_{eq}^{-1}}{1-\left(\frac{\omega}{\omega_0}\right)^2 + j\frac{\omega}{Q\omega_0}}}$$

Lowpass Biquad



- Go through pages 11-22 of Module 11