Lecture 19: Capacitive Transducers

- Announcements:
- I am traveling today, so this is a recorded video lecture
- Midterm Exam this coming Thursday, March 23, 3:30-5 p.m., 521 Cory (right here)
- HW#5 will go online soon, if not already
  - Will be due well after Spring Break
  - The idea is that you need not work on it during Spring Break, but you can if want
- If you haven't already done so, pick up last past midterm solutions in the box outside my office
- Module 12 on Capacitive Transducers online
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- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
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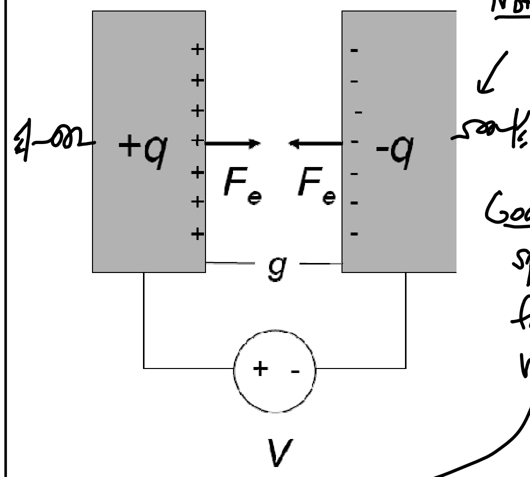
- Last Time:
- Specified circuit model for mechanical behavior
- Must still develop a circuit model for the electrical-to-mechanical transducer

Basic Physics of Electrostatic Actuation



Note: Assume the plates are supported elastically → can move

Goal: Determine gap spacing g as a function of input variables

1st: Determine the energy of the system.

2nd: Ask: What can I do to Δ the energy of the system?

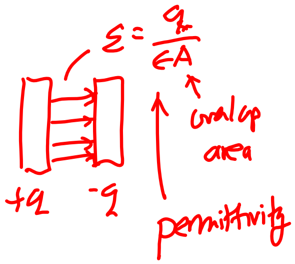
① change the charge q

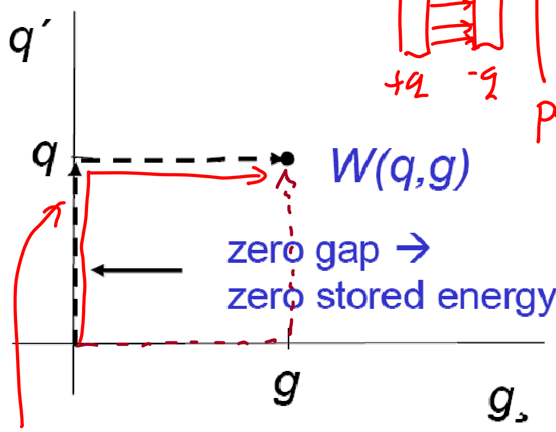
② change the separation g

$$\Delta W(q,g) = V\Delta q + F_e \Delta g$$

$$dW = V dq + F_e dg$$

Stored Energy





No change in charge: $dq = 0$

$$W = 0 + \int_0^g F_e \, dg'$$

$$F_e = \left(\frac{q}{2}\right)\mathcal{E} = \frac{1}{2}\frac{q^2}{\epsilon A} \quad \text{(independent of } g\text{)}$$

$$\therefore W = \int_0^g F_e \, dg' = F_e g' \Big|_0^g = F_e g$$

$$\boxed{W(g) = \frac{1}{2}\frac{q^2}{\epsilon A} g}$$

= **

over ⟹ *→

*→ Work done to charge C to q at a fixed gap g:

$$dW = V\,dq + F_e\,\cancel{dg}^{\,0}$$

For a capacitor:

$$q = CV \rightarrow V = \frac{q}{C}$$

$$\therefore W(q) = \int_0^q V\,dq' = \int_0^q \left(\frac{q'}{C}\right) dq' = \frac{1}{2}\frac{q^2}{C}$$

$$\boxed{= \frac{1}{2}\frac{q^2}{\epsilon A} g = W(q)}$$

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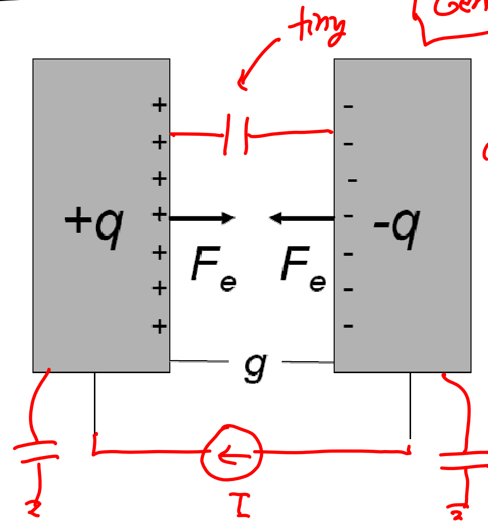
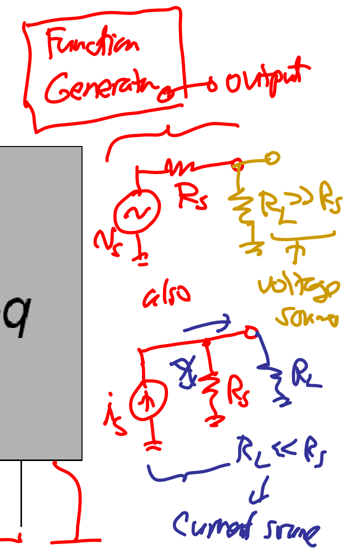
Charge Control Case





From $dW = V\cancel{dq} + F_e dg$

⇒ Force is given by

$$F_e = \left.\frac{\partial W(q,g)}{\partial g}\right|_{q=\text{const.}} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A} g\right)$$

$$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}} \Rightarrow \text{indep. of gap spacing!}$$

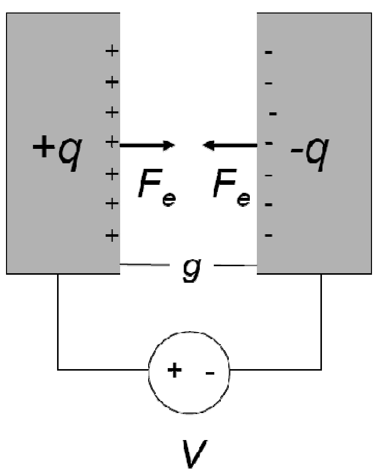
⇒ voltage is given by:

$$V = \left.\frac{\partial W(q,g)}{\partial q}\right|_{g=\text{const.}} = \frac{\partial}{\partial q}\left(\frac{1}{2}\frac{q^2}{\epsilon A} g\right)$$

$$= \frac{qg}{\epsilon A} \rightarrow \boxed{V = \frac{q}{C}} \checkmark$$

(consistent w/ what we know)

$\boxed{\text{Voltage Control}}$



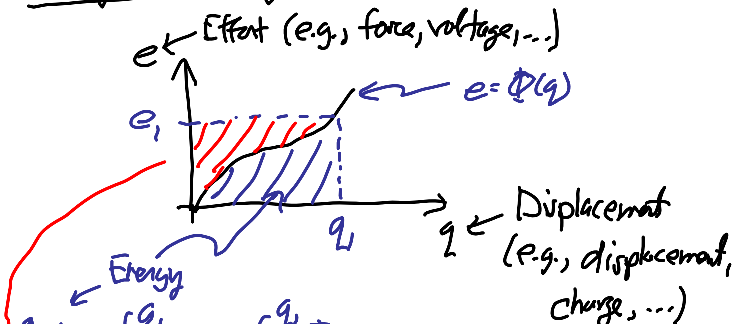
Want to write:
$F_e = f(V)$

We know this:
$dW = Vdq + F_e dg$
$W = W(q,g)$

Need: $W'(V,g)$ ← replace charge q w/ voltage V

Can get this using a Legendre transformation.

Energy & Co-Energy



$$W(q_1) = \int_0^{q_1} e\, dq = \int_0^{q_1} \Phi(q)\, dq$$

Co-Energy:

$$W'(e_1) = \int_0^{e_1} q\, de = \int_0^{e_1} \Phi^{-1}(e)\, de$$

For a linear system, these will be equal.

Can define co-energy as:

$$\underset{\text{co-energy}}{W'(e)} = eq - \underset{\text{energy}}{W(q)} \quad \text{(from the plot)}$$

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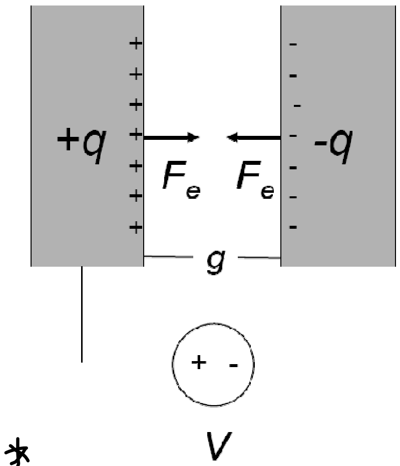
Co-Energy Formulation for Voltage Control



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$$W'(V,g) = Vq - W(q,g)$$

Differentially, this becomes

$$dW'(V,g) = (q\,dV + \cancel{V\,dq}) - dW(q,g)$$

$$\left[dW(q,g) = F_e\,dg + \cancel{V\,dq} \right]$$

$$\boxed{dW'(V,g) = q\,dV - F_e\,dg}$$ ← working co-energy expression

Find co-energy in terms of voltage, V:

$$W' = \int_0^V q(g,V')\,dV' = \int_0^V \left(\frac{\epsilon A}{g}\right) V'\,dV'$$

$$= \frac{1}{2}\left(\frac{\epsilon A}{g}\right)V^2 = \frac{1}{2}CV^2 \checkmark$$ (as expected)

Voltage-Controlled Electrostatic Force:

$$F_e = -\left.\frac{\partial W'(V,g)}{\partial g}\right|_{V=const.}$$

$$= -\frac{1}{2}\left(\frac{\epsilon A}{g^2}\right)V^2 = \boxed{\frac{1}{2}\frac{C}{g}V^2 = F_e}$$

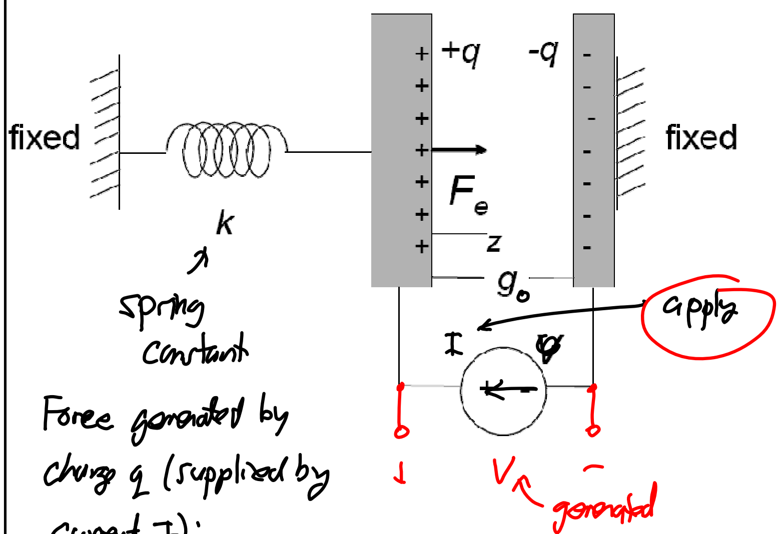
depends on gap!

Charge:

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_{g=const.} = \frac{\epsilon A}{g}V = CV \checkmark$$

(as expected)

Charge-Control of a Spring-Suspended C



spring constant

Force generated by charge q (supplied by current I):

$$F_e = \frac{\partial W(q,g)}{\partial g}\bigg|_q = \frac{q^2}{2\epsilon A}$$

Restoring force of spring: $F_{spring} = kz = F_e$ ← equilibrium

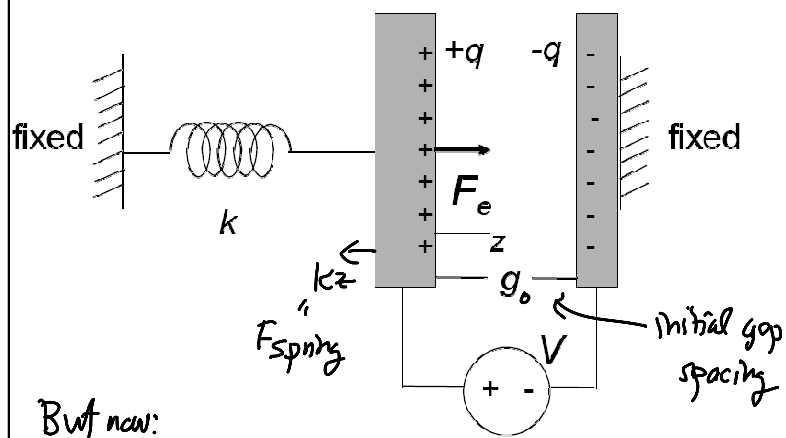
The gap:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

↳ q↑ can drive $g \to 0$ in continuous fashion

$$V = \frac{q}{C} = \frac{q}{\epsilon A} g = \boxed{\frac{q}{\epsilon A}\left(g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V}$$ ← V↓ as g↓

Voltage-Control of a Suspended C



But now:

$$F_e = \frac{\partial W'(V,g)}{\partial g}\bigg|_q \rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

g shows up on both sides!

If V↑ → g↓ → F_e↑ (+) Feedback!

If loop gain > 1, then this will go unstable!

plate will collapse! (into the electrode)

Charge: (for a stable gap)

$$q = \frac{\partial \mathcal{W}'(V,g)}{\partial V}\bigg|_g = CV \checkmark \quad \text{(as expected)}$$

Stability Analysis

⇒ determine under what conditions voltage control will cause collapse of the plates:

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_o - g)}_{spring}$$

What happens when I change g by a small increment dg?

get an increment in the net attractive force F_{net}

$$\underbrace{dF_{net}}_{(-)} = \frac{\partial F_{net}}{\partial g} dg = \left[-\frac{\epsilon A V^2}{g^3} + k\right] \underbrace{dg}_{(-)}$$

If $g\downarrow \to dg = (-)$, then for stability need $F_{net}\downarrow \to dF_{net} = (-)$

This must be (+)! → otherwise, the plates collapse!

Thus: $\boxed{k > \frac{\epsilon A V^2}{g^3}}$ (for a stable uncollapsed system)

Pull-in Voltage V_{PI} & Pull-in Gap g_{PI}

$V_{PI} \triangleq$ voltage @ which plates collapse

$g_{PI} \triangleq$ gap @ " " "

The plates go unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \quad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_o - g_{PI}) \quad (2)$$

Substitute (1) into (2):

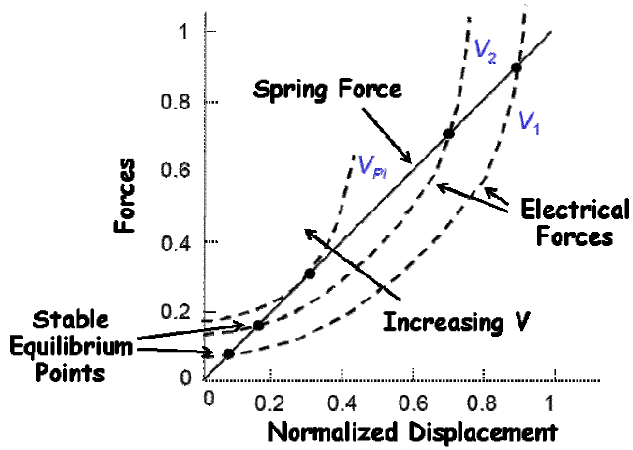
$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2\cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^3}(g_o - g_{PI})$$

$$\frac{g_o - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_o = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_o}$$

When the gap is driven by a voltage to (2/3) the initial gap → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow \boxed{V_{PI} = \sqrt{\frac{8}{27}\frac{k g_o^3}{\epsilon A}}}$$



Advantages of Electrostatic Actuators:

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale