Lecture 12: Beam Bending

- Announcements:
- HW#3 due Tuesday, 3/10, at 8 a.m.
- Module 8 on "Microstructural Elements" online
- Midterm less than 3 weeks away
- ----------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- ----------------------------------
- Last Time:
- Looking at forces & moments in equilibrium
- Now, move on to bent beam analysis …

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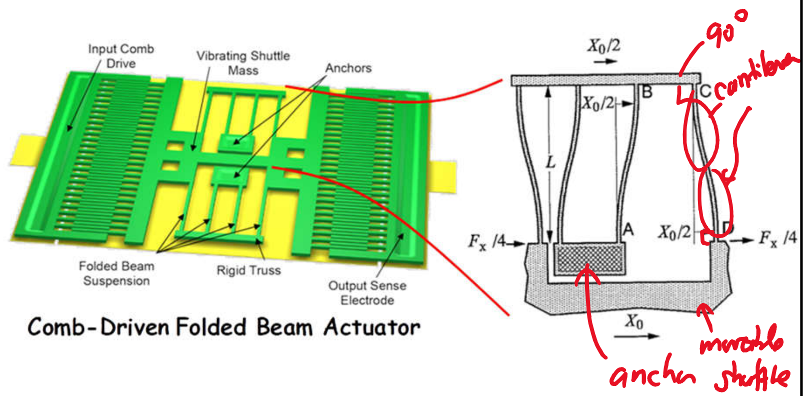
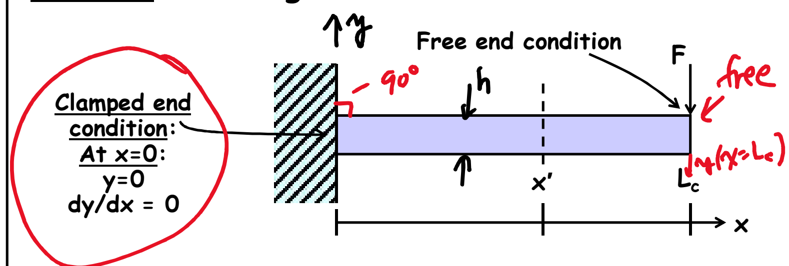
- Springs and suspensions very common in MEMS
- Coils are popular in the macro-world; but not easy to make in the micro-world
- Beams: simpler to fabricate and analyze; become "stronger" on the micro-scale → use beams for MEMS



Comb-Driven Folded Beam Actuator

Problem: Bending a Cantilever Beam



- Objective: Find relation between tip deflection $y(x=L_c)$ and applied load F
- Assumptions:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible

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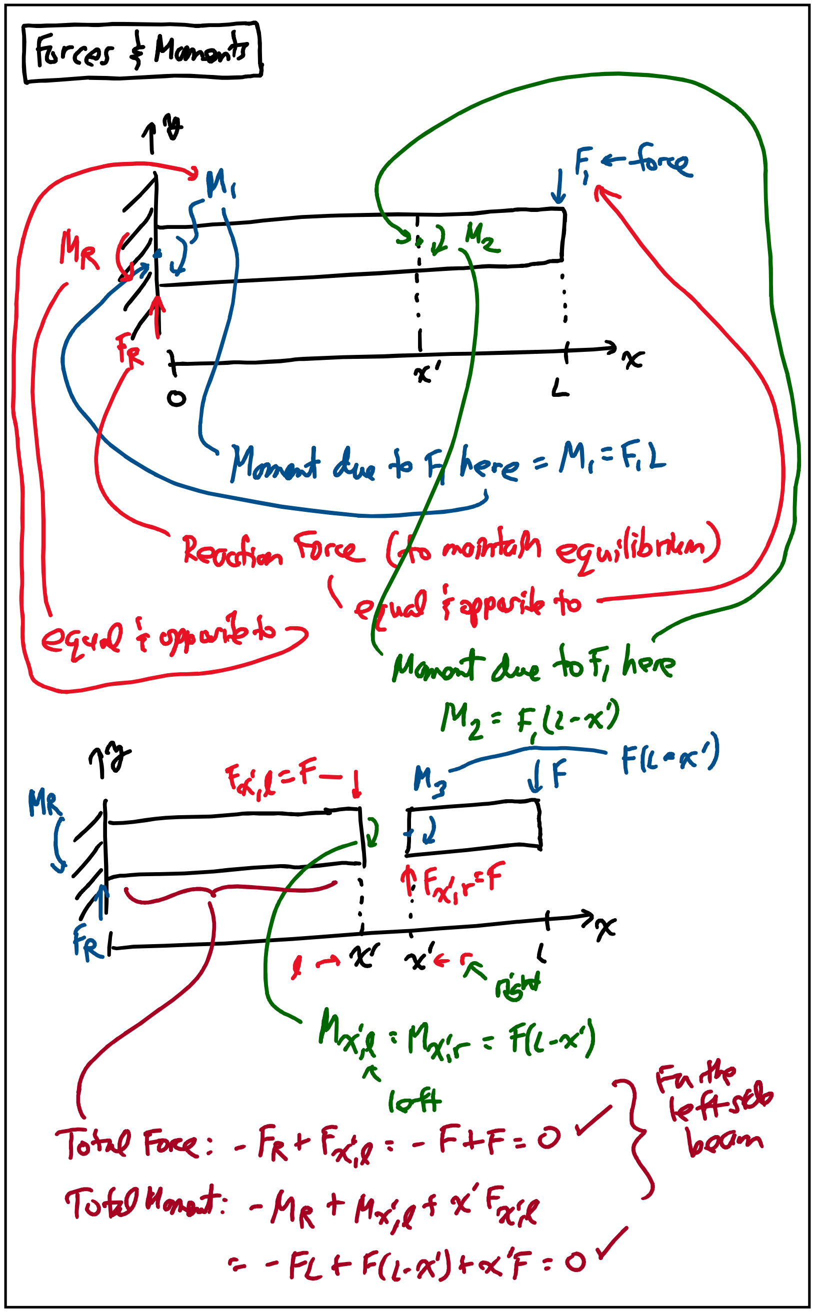
Forces & Moments
y
M_1
F_1 ← force
M_R
M_2
F_R
0
x'
L
x
Moment due to F_1 here = $M_1 = F_1 L$
Reaction Force (to maintain equilibrium)
equal & opposite to
equal & opposite to
Moment due to F_1 here
$M_2 = F_1(L-x')$
$F_{x',l} = F$
M_3
F
$F(L-x')$
M_R
$F_{x',r} = F$
F_R
$l \to x'$
$x' \leftarrow r$
right
$M_{x',l} = M_{x',r} = F(L-x')$
left
Total Force: $-F_R + F_{x',l} = -F + F = 0$ ✓
Total Moment: $-M_R + M_{x',l} + x' F_{x',l}$
$= -FL + F(L-x') + x'F = 0$ ✓
For the left-side beam

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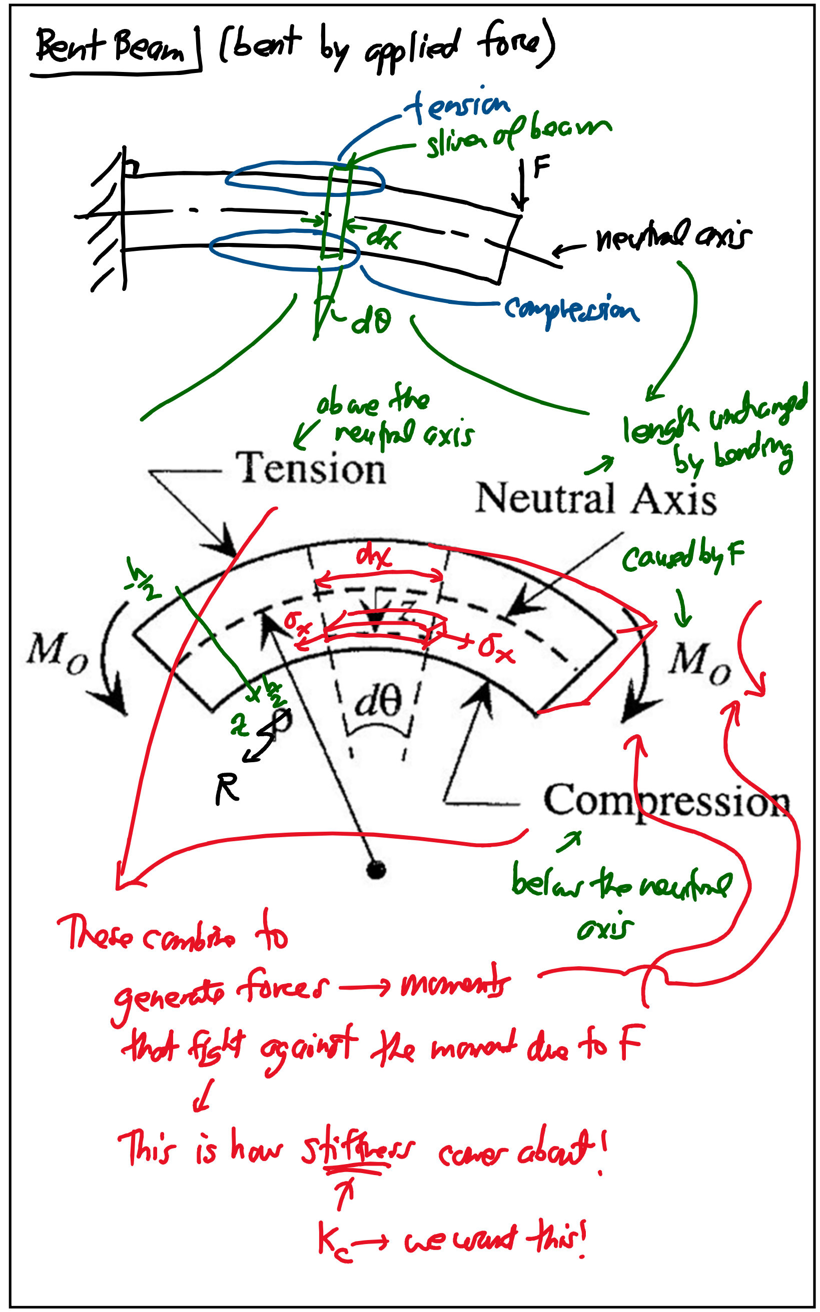
Bent Beam (bent by applied force)
tension
sliver of beam
F
dx
neutral axis
compression
$d\theta$
above the neutral axis
length unchanged by bending
Tension
Neutral Axis
caused by F
dx
$h/2$
σ_x
σ_x
z
M_O
M_O
$d\theta$
R
Compression
below the neutral axis
These combine to
generate forces → moments
that fight against the moment due to F
This is how stiffness comes about!
k_c → we want this!

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Beam Segment in Pure Bending

⇒ consider the segment bounded by dashed lines defining $d\theta$

At $z=0$: neutral axis → segment length $= dx = R\,d\theta$ (1)

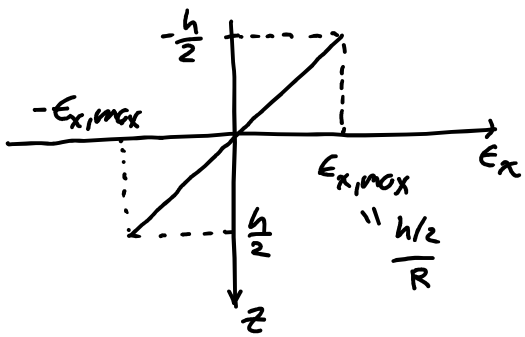
At any z: segment length $= dL = (R-z)\,d\theta$ (2)

Combine (1) & (2): $dL = dx - z\,d\theta = dx - \frac{z}{R}dx$

Thus, the axial strain @ z:

$$\epsilon_x = \frac{dL-dx}{dx} = -\frac{z}{R} \Rightarrow \boxed{\epsilon_x = -\frac{z}{R}}$$

Thus, the strain varies linearly along the beam thickness:



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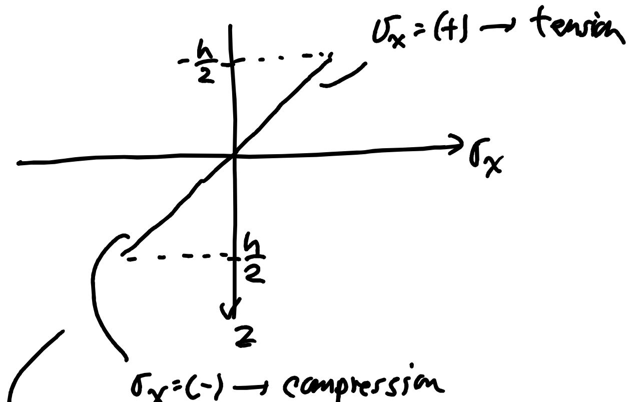
Of course, there is a corresponding axial stress:

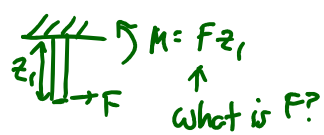
$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$

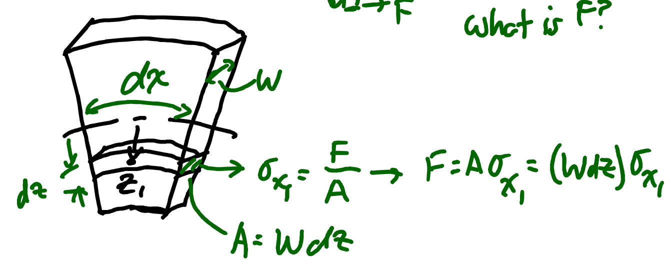


This gradient of stress then generates $\underbrace{\text{a bending moment!}}$

in response to the original applied moment (from F)

Stress → Force:





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$\Rightarrow$ integrate stress through the thickness of the beam:

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} \underbrace{[(W dz)\sigma_x]}_{\text{force}} \cdot z$$

$$= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{EWz^2}{R} dz \Rightarrow \boxed{M = -\left(\frac{1}{12} W h^3\right)\frac{E}{R}}$$

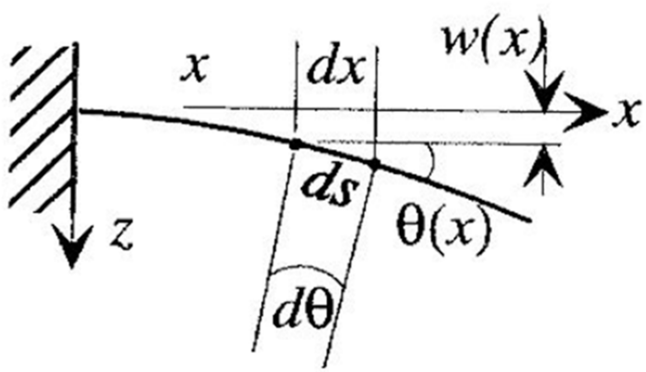
$\left[\sigma_x = -\frac{zE}{R}\right]$

$\frac{1}{12} W h^3 = I \triangleq$ Moment of Inertia

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

Note: (+) radius of curvature → (−) internal bending moment

Differential Equation for Beam Bending



Write out some geometric relationships:

$\Rightarrow$ then use small angle approx:

$$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \rightarrow ds \simeq dx$$

$$\tan\theta = \frac{dw}{dx} = \text{slope of the beam @ this point} \rightarrow \theta \simeq \frac{dw}{dx} \quad (1)$$

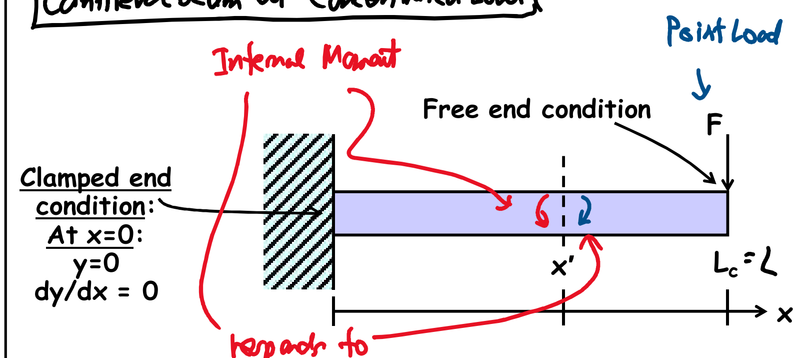
$$ds = R\,d\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \longrightarrow \frac{1}{R} = \frac{d\theta}{dx} \quad (2)$$

Insert (1) into (2):

$$\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$$

Diff. Eqn. for Small Angle Beam Bending

Cantilever Beam w/ Concentrated Load



Internal Moment @ position x: $M = -F(L-x)$

Thus: $\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$

w/ Clamped End B.C.: $w(x=0)=0$, $\frac{dw}{dx}(x=0)=0$

Free End B.C.: none

Solve to get w:

⇒ We Laplace a trial solution:

$w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s

$$\boxed{w = \frac{FL}{2EI}x^2\left(1-\frac{x}{3L}\right)}$$

deflection @ x due to a point load F applied @ $x=L$

Maximum Deflection → occurs @ $x=L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \longrightarrow F = \underbrace{\left(\frac{3EI}{L^3}\right)}_{\text{stiffness} \triangleq k_c \text{ @ } x=L} w(x=L) = k_c w(x=L)$$

where $k_c = \frac{3EI}{L^3} \triangleq$

$$\left[I = \tfrac{1}{12}Wh^3\right] \Rightarrow \boxed{k_c = \frac{1}{4}EW\frac{h^3}{L^3}}$$

Ex. $L = 100\mu m$, $W = 2\mu$, $h = 2\mu m$

polysilicon → $E = 150$ GPa

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = \underline{\underline{0.6\ N/m}}$$

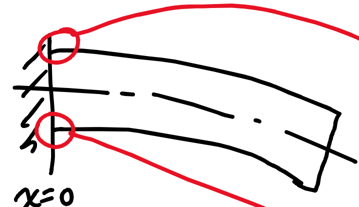
Maximum Stress in a Bent Cantilever

From before, the radius of curvature is given by:

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

⇒ R minimizes when $\frac{1}{R}$ maximizes when $x=0$:

$$[x=0] \Rightarrow \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$$

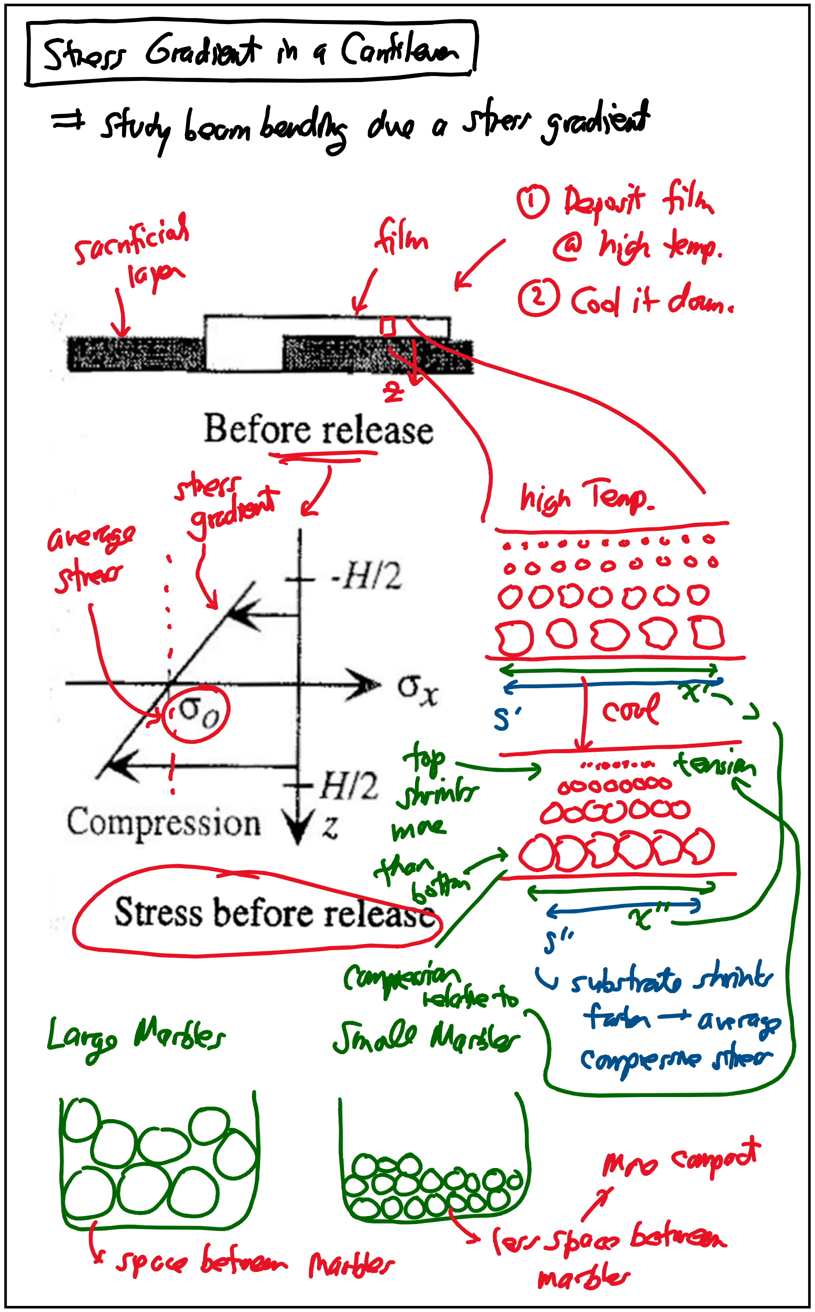


Strain maximizes:

1. At top surface → tensile
2. At bottom surface → compression

$$\epsilon_{max} = \frac{z}{R} = \frac{h}{2}\frac{1}{R} = \boxed{\frac{h}{2}\frac{FL}{EI} = \epsilon_{max}}$$

$$\left[I = \tfrac{1}{12}Wh^3\right] \Rightarrow \boxed{\epsilon_{max} = \frac{6L}{EWh^2}F}$$


Stress Gradient in a Cantilever
⇒ study beam bending due a stress gradient
sacrificial layer
film
① Deposit film @ high temp.
② Cool it down.
z
Before release
stress gradient
average stress
-H/2
σx
σ0
H/2
Compression
z
Stress before release
high Temp.
s'
x'
cool
top shrinks more than bottom
tension
x''
s''
compression relative to
substrate shrinks faster → average compressive stress
Large Marbles
Small Marbles
space between marbles
more compact
less space between marbles

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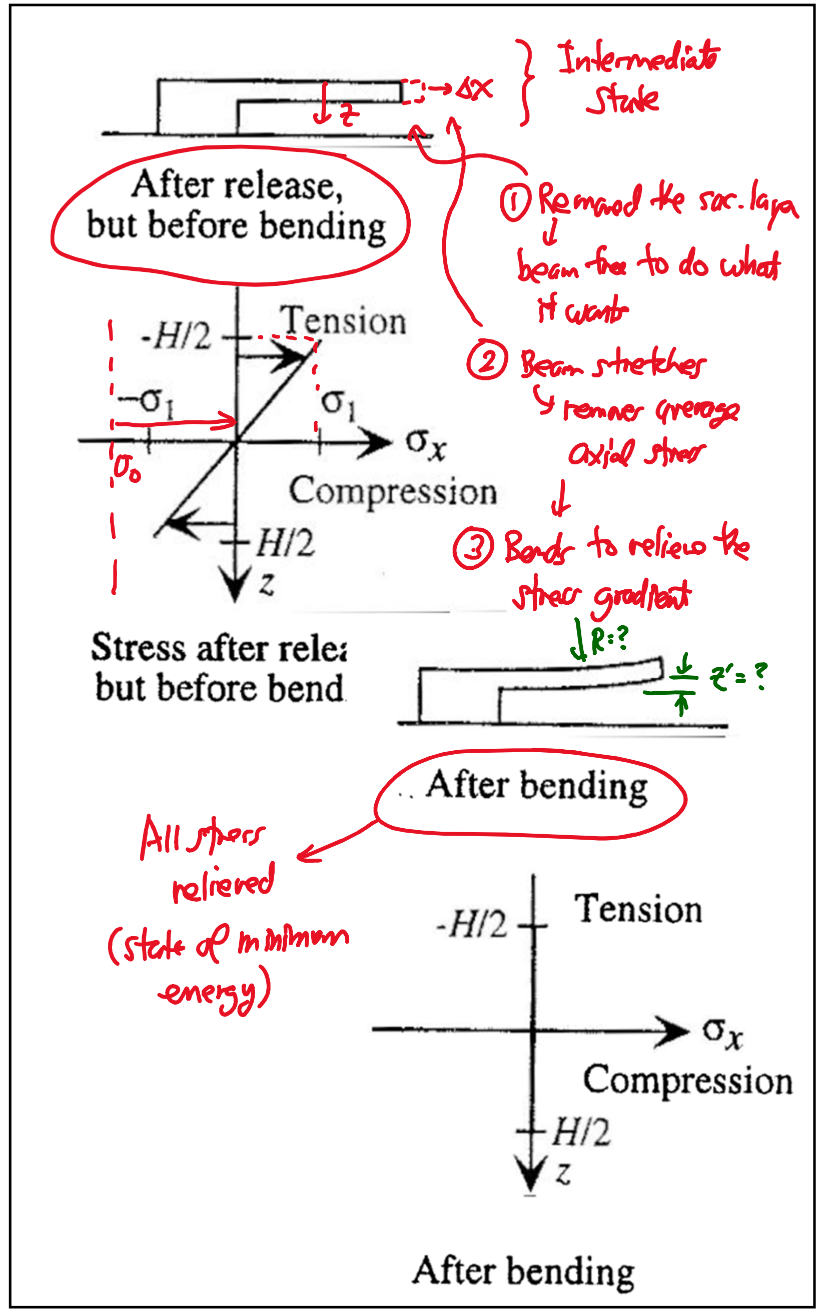
Intermediate State
Δx
z
After release, but before bending
① Removed the sac. layer
beam free to do what it wants
② Beam stretches
remove average axial stress
③ Bends to relieve the stress gradient
R=?
z'=?
-H/2
Tension
-σ1
σ1
σx
σo
Compression
H/2
z
Stress after relea
but before bend
After bending
All stress relieved
(state of minimum energy)
-H/2
Tension
σx
Compression
H/2
z
After bending

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Bending Due to Stress/Strain Gradients

Find the radius of curvature:

Prior to release, axial stress: $\sigma = \sigma_0 - \frac{\sigma_1}{(H/2)} z$

The internal moment:

$$M_x = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left[(W \cdot dz)\sigma\right] \cdot z = \int_{-\frac{H}{2}}^{\frac{H}{2}} W\left(z\sigma_0 - \frac{\sigma_1 z^2}{(H/2)}\right) dz$$

$$= W\left(\frac{1}{2}\sigma_0 z^2 - \frac{2\sigma_1 z^3}{3H}\right)\Bigg|_{-H/2}^{H/2}$$

$$= W\left(\cancel{\frac{1}{2}\sigma_0 \frac{H^2}{4}} - \frac{2}{3}\sigma_1 \frac{H^2}{8} - \cancel{\frac{1}{2}\sigma_0 \frac{H^2}{4}} - \frac{2}{3}\sigma_1 \frac{H^2}{8}\right)$$

average stress cancels out

$$\boxed{M_x = -\frac{1}{6}\sigma_1 W H^2}$$

Thus, the radius of curvature:

$$\frac{1}{R} = -\frac{M_x}{E'I} \rightarrow R = \frac{E'I}{M_x} = \frac{1}{2}\frac{E'H}{\sigma_1}$$

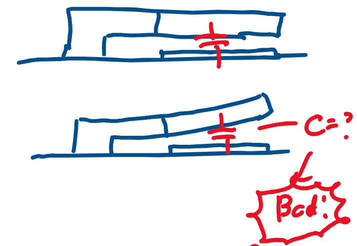
$\left[I = \frac{1}{12} W h^3\right]$

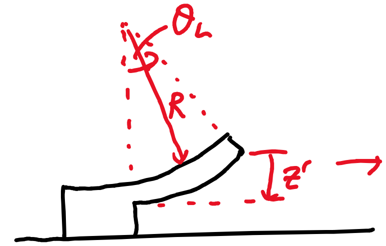
Biaxial Modulus (if W is large)

$$\boxed{R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1}}$$

Radius of Curvature for a Cantilever w/ a Stress Gradient

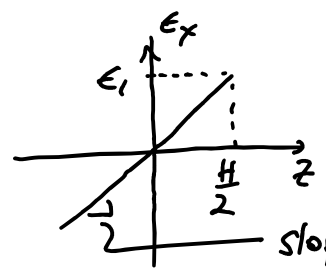
Radius of Curvature → z'





→ integrate over θ_L to get z'
↓
Do this in homework ...

Definition. Strain Gradient



slope ≜ Strain Gradient = Γ

$$R = \frac{1}{2}\frac{E}{1-\nu}\frac{H}{\sigma_1} = \frac{H}{2}\frac{E'}{\sigma_1} = \frac{(H/2)}{\epsilon_1} = \frac{1}{\Gamma} \rightarrow \boxed{\Gamma = \frac{1}{R}}$$ ✓