## Lecture 14: Beam Combos II

- Announcements:
- HW#4 online soon, due Tuesday, 3/17, at 8 a.m.
- Midterm Exam: Thursday, March 19, 9:30-11:00 a.m., 521 Cory (right here)
- UC Berkeley has stopped ground classes in an effort to suppress Coronavirus
- This is a video-recorded lecture, as will be subsequent lectures until the university goes back to ground classes
- Office hours are going to Zoom per my recent Piazza post
- ------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
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- Last Time:
- Finished beam combinations and mechanical spring circuits, using a mechanical filter example
- Now, continue with examples and methods for handling stress in beam combos

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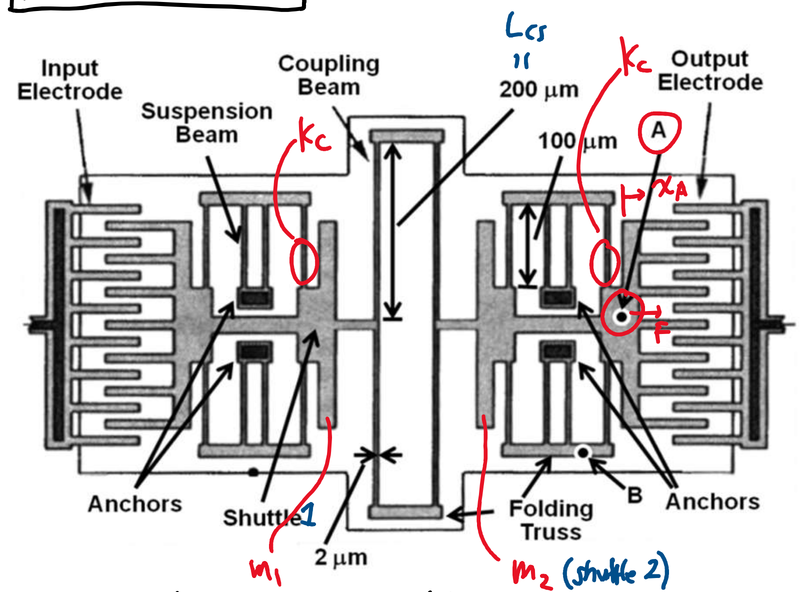
## Micromechanical Filter

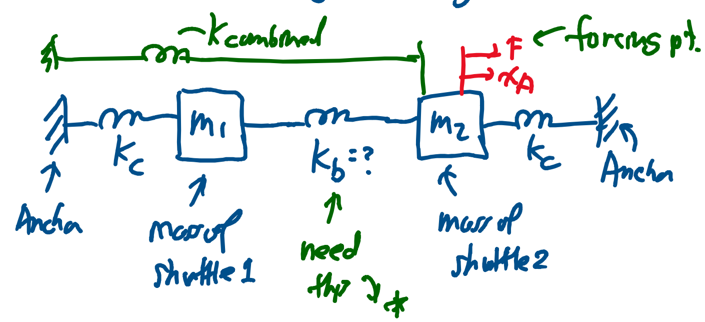



⇒ Find the stiffness at point A.

apply F@A → what is $x_A$? } $x_A = \frac{F}{k_A}$

$k_A \triangleq$ stiffness @ point A ← want this

Assume: shuttles & folding trusses are rigid, beams are small ∴ neglect their mass

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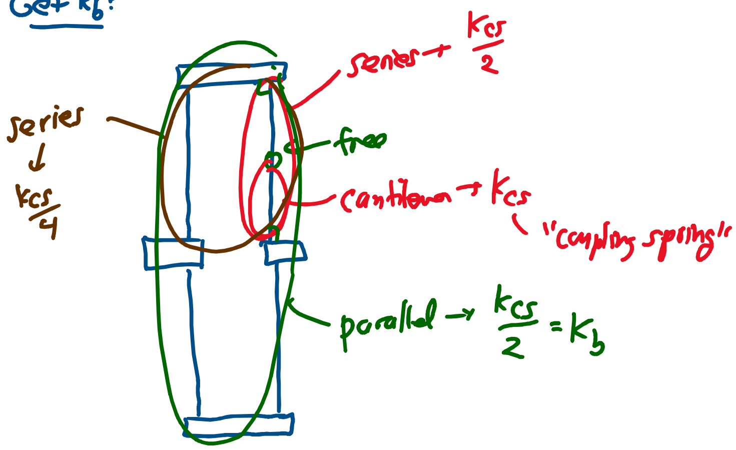
Get $k_b$:




$\therefore\ k_A = k_c + k_{combined}$

$= k_c + k_c \| k_b = k_c + k_c \| \frac{k_{cs}}{2} = k_A$

where $k_c = \frac{24EI_z}{L^3}$

$k_{cs} = \frac{24EI_z}{L_{cs}^3}$



## Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$

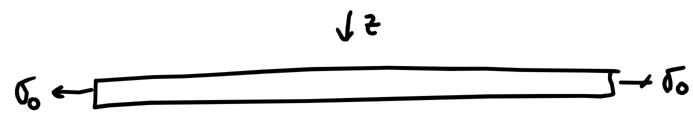

Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4y}{dx^4} - S\frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

Axial Load

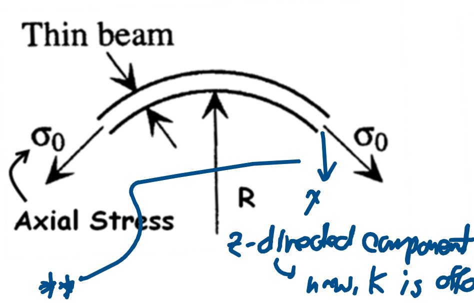
## Heuristic Derivation for the Euler Beam Equation

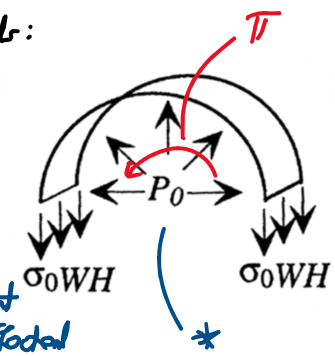
Consider first a straight beam under an axial stress:




⇒ no effect on z-directed stiffness when the beam is straight

... but when the beam bends:



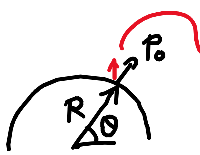
* Upward pressure $P_o$ to counteract the downward force from ** to keep everything in static equilibrium

For ease of analysis:

Assume the beam bends to an angle $\pi$

↳ Downward vertical force: $2\sigma_o WH$

Get upward force due to $P_o$:




$P_z(\theta) = P_o \sin\theta$

$$F_u = \int_0^{\pi} (P_o \sin\theta) W (R d\theta)$$

$$= -P_o W R \cos\theta \Big|_0^{\pi}$$

$$= 2RWP_o$$

[Equilibrium] → $2RWP_o = 2\sigma_o WH \longrightarrow P_o = \dfrac{\sigma_o H}{R}$

$$\left[ q_o = \frac{\text{beam load}}{\text{unit length}} = P_o W, \quad \frac{1}{R} = \frac{d^2 w}{dx^2} \right]$$ beam displacement

$$q_o = \sigma_o W H \frac{d^2 w}{dx^2}$$ ← generalizes to the case of small displacements & angles

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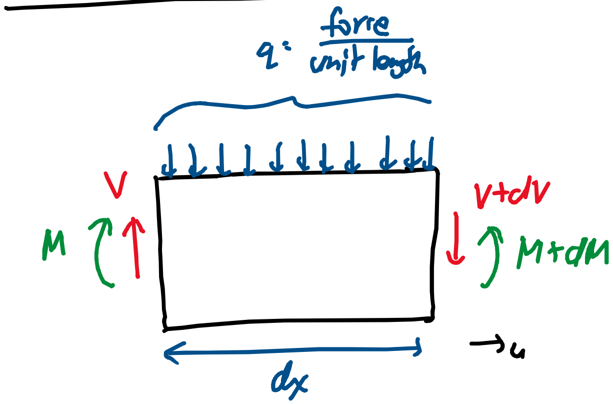
**Using the Differential Beam Bending Eq.]**

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \xrightarrow[???]{} \frac{d^4 w}{dx^4} = \frac{q}{EI}$$ ← load/unit length

+ ↙ +

* **Relationships Between Forces & Moments on a Fully-Loaded Differential Beam Element**




[Total Static Equilibrium] ⟹ total force = 0

$F_T$ = total force = $q\,dx + (\cancel{V} + dV) - \cancel{V} = 0$

$$\therefore \boxed{\frac{dV}{dx} = -q} \quad (1)$$

⇒ also, total moment w/r to left-hand edge = 0

$$M_T = (\cancel{M} + dM) - \cancel{M} - (V + \cancel{dV})dx - \tfrac{1}{2} q \cancel{dx^2} = 0$$

[neglect products of differentials]

$$\int_0^{dx} (q\,du)u = \tfrac{1}{2} q\,dx^2$$

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$dM - V\,dx = 0 \rightarrow \boxed{\frac{dM}{dx} = V}$ (2)

Using (1) & (2):

$$\left[\frac{d^2M}{dx^2} = \frac{dV}{dx} = -q\right]$$ +

++

$$EI\frac{d^4w}{dx^4} = q + q_0$$ +++

external load

equiv. load from axial stress

$$\left[q_0 = \sigma_0 WH \frac{d^2w}{dx^2}\right]$$

$$\boxed{EI\frac{d^4w}{dx^2} - \underbrace{(\sigma_0 WH)}_{\text{tension in beam} = S}\frac{d^2w}{dx^2} = q}$$

force

Euler Beam Equation

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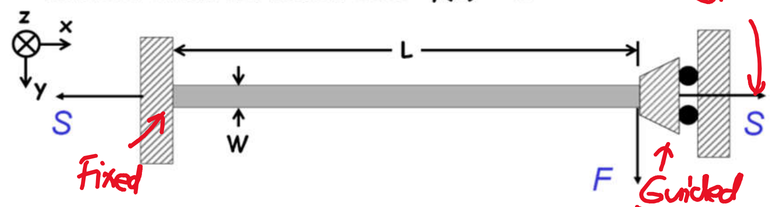
## Clamped-Guided Beam Under Axial Load

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$

Stress




Governing differential equation: (Euler Beam Equation)

$$EI_z\frac{d^4y}{dx^4} - S\frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

Force

Axial Load

Need to solve this, then find the stiffness against this force (@ this location)

- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko: (Note: These include both bending & stretching)

$S > 0$ (tension) $$k^{-1} = \frac{pL - 2\tanh(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

$S < 0$ (compression)

$$k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

where $p = \sqrt{\frac{|S|}{EI_z}}$

To use, must know

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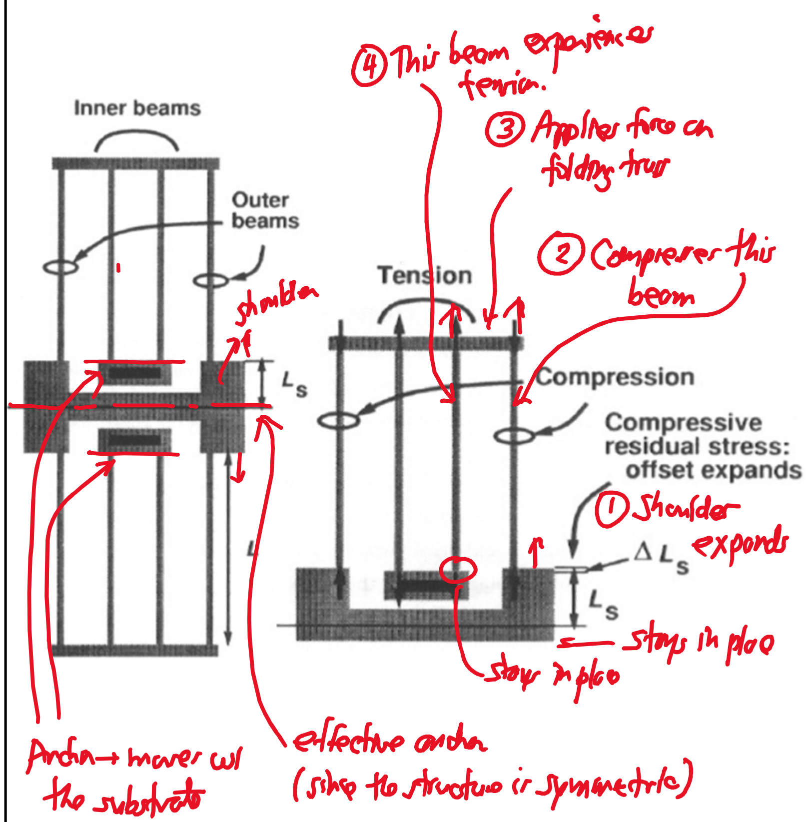
## Folded Beams Are Not Perfect




Get S:

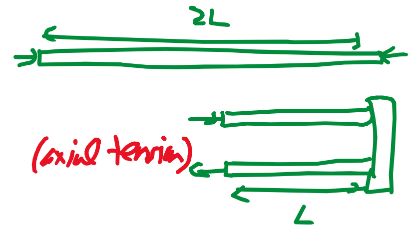
① If the polySi structural material strain is $\epsilon_r$, then the shoulder expands $\Delta L_s = \epsilon_r L_s$

② This then applies a load to the beams, $\Delta L = \Delta L_s$.



③ Beam Stress:

$$\epsilon_b = \frac{\Delta L}{2L} = \frac{\Delta L_s}{2L} = \pm \epsilon_r \frac{L_s}{2L}$$



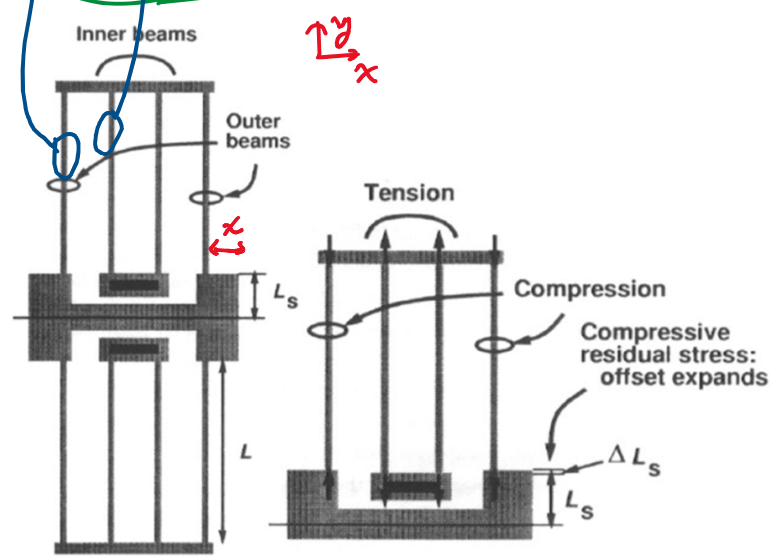

Stress Force:

$$S = \pm E \epsilon_r \left(\frac{L_s}{2L}\right) W h \quad \text{(axial tension)}$$

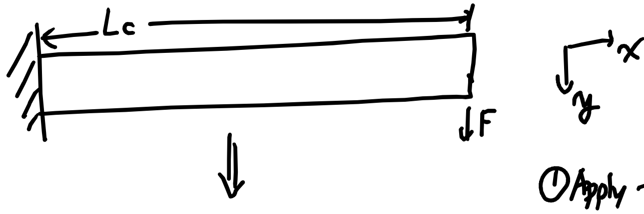
④ Spring Constants: 4 in parallel — series

$$k = 4\left(k_{com}^{-1} + k_{ten}^{-1}\right)^{-1} \qquad : k_{com} \,\|\, k_{ten}$$

$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{p|S|} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$



Same Problem as Before: Take a beam, apply a force.




① Apply force.


② Beam responds by bending.

③ This force has done work:

$W = F \cdot y(L_c)$

④ Strain generated.

So the beam has received an influx of stored energy.

magnitude of " " determined by shape.

⑤ Then

$U = \text{Stored Energy} - \text{Work Done} \rightarrow 0$

When we choose the right shape.

transfer function

$y(x) = f(x)$

This is how we get the beam's response to F!

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