Shear Strain Energy

$$\mathcal{W}_{shear} = \frac{3(EI_z)^2}{4GWh}\int_0^L\left(\frac{d^3y}{dx^3}\right)^2 dx$$

Shear Modulus

- See W.C. Albert, "Vibrating Quartz Crystal Beam Accelerometer," Proc. ISA Int. Instrumentation Symp., May 1982, pp. 33-44

Applying the Principle of Virtual Work

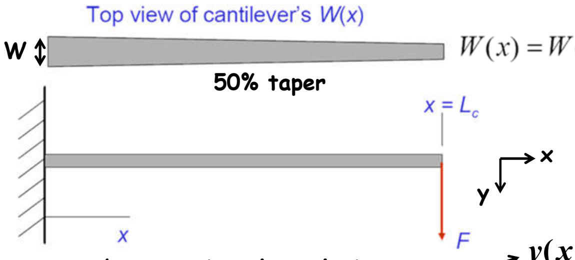
- **Basic Procedure**:
  - Guess the form of the beam deflection under the applied loads
  - Vary the parameters in the beam deflection function in order to minimize:

$$U = \overbrace{\sum_j \mathcal{W}_j}^{\text{Sum strain energies}} - \sum_i \overset{\text{Assumes point load}}{F_i}\underset{\text{Displacement at point load}}{u_i}$$

  - Find minima by simply setting derivatives to zero
- See Senturia, pg. 244, for a general expression with distrubuted surface loads and body forces

Example: Tapered Cantilever Beam

- **Objective**: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



$$W(x) = W\left(1-\frac{x}{2L_c}\right)$$

Adjustable parameters: minimize U

- Start by guessing the solution $\longrightarrow y(x) = c_2x^2 + c_3x^3$
  - It should satisfy the boundary conditions
  - The strain energy integrals shouldn't be too tedious
    - This might not matter much these days, though, since one could just use matlab or mathematica

Strain Energy And Work By F

$$U = \mathcal{W}_{bend} - F\cdot y(L_c)$$

$$\mathcal{W}_{bend} = \frac{1}{2}E\int_0^{L_c} I_z(x)\left(\frac{d^2y}{dx^2}\right)^2 dx$$

(Bending Energy)

$$I_z(x) = \frac{W(x)h^3}{12}$$

$$\frac{d^2y}{dx^2} = 2c_2 + 6c_3x$$

(Using our guess)

$$W(x) = W\left(1-\frac{x}{2L_c}\right)$$

$$= \frac{1}{24}EWh^3\int_0^{L_c}\left(1-\frac{x}{2L_c}\right)(2c_2+6c_3x)^2dx - F\overbrace{\left(c_2L_c^{\;2}+c_3L_c^{\;3}\right)}^{\text{Tip Deflection}}$$

Find c_2 and c_3 That Minimize U

- Minimize U → basically, find the c_2 and c_3 that brings U closest to zero (which is what it would be if we had guessed correctly)
- The c_2 and c_3 that minimize U are the ones for which the partial derivatives of U with respective to them are zero:

$$\frac{\partial U}{\partial c_2}=0 \qquad \frac{\partial U}{\partial c_3}=0$$

- Proceed:
  - ➯ First, evaluate the integral to get an expression for U:

$$U=EWh^3\left\{\frac{5c_3^{\,2}}{16}L_c^{\,3}+\frac{c_2c_3}{3}L_c^{\,2}+\frac{c_2^{\,2}}{8}L_c\right\}-F\left(c_2L_c^{\,2}+c_3L_c^{\,3}\right)$$

Minimize U (cont)

- Evaluate the derivatives and set to zero:

$$\frac{\partial U}{\partial c_2}=0=\left(\frac{EWh^3}{3}c_3-F\right)L_c^{\,2}+\left(\frac{EWh^3}{4}c_2\right)L_c$$

$$\frac{\partial U}{\partial c_3}=0=\left(\frac{5}{8}EWh^3c_3-F\right)L_c^{\,3}+\left(\frac{EWh^3}{3}c_2\right)L_c^{\,2}$$

- Solve the simultaneous equations to get c_2 and c_3:

$$\boxed{c_2=\left(\frac{84}{13}\right)\frac{FL_c}{EWh^3} \qquad c_3=-\left(\frac{24}{13}\right)\frac{F}{EWh^3}}$$

The Virtual Work-Derived Solution

- And the solution:

$$y(x)=\left(\frac{24F}{13EWh^3}\right)\left(\left(\frac{7}{2}\right)L_c-x\right)x^2$$

- Solve for tip deflection and obtain the spring constant:

$$y(L_c)=\left(\frac{24F}{13EWh^3}\right)\left(\frac{5}{2}\right)L_c^{\,3} \qquad k_c=F/y(L_c)=\left(\frac{13EWh^3}{60L_c^{\,3}}\right)$$

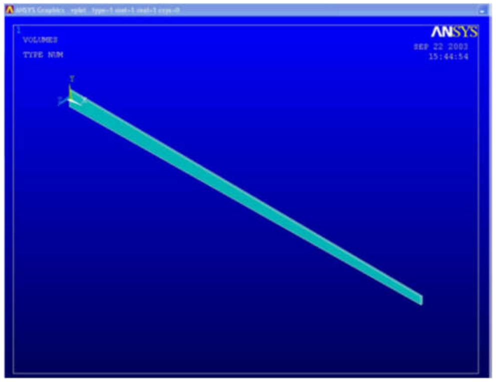
- Compare with previous solution for constant-width cantilever beam (using Euler theory):

$$y(L_c)=\left(\frac{4F}{EWh^3}\right)L_c^{\,3} \longrightarrow$$ 13% smaller than tapered-width case

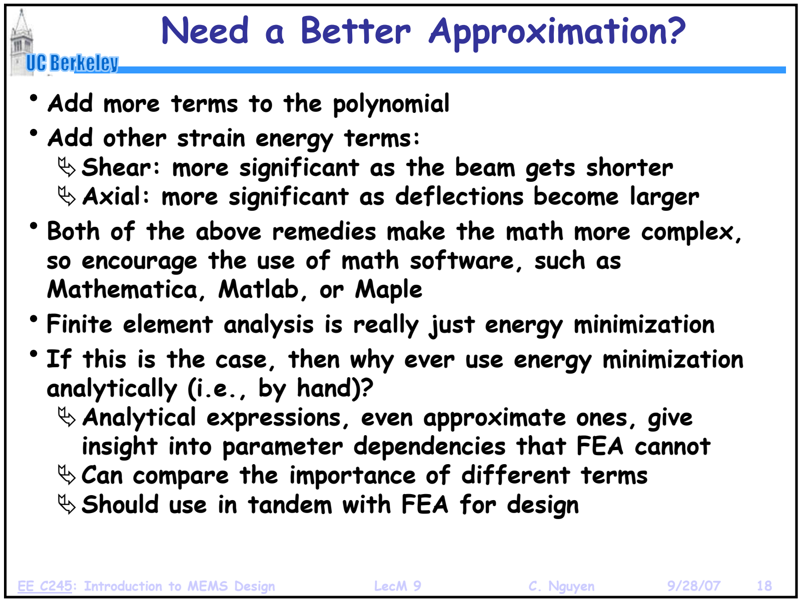
Comparison With Finite Element Simulation

- **Below**: ANSYS finite element model with

L = 500 μm W_{base} = 20 μm E = 170 GPa
h = 2 μm W_{tip} = 10 μm



- **Result**: (from static analysis)
  - ➯ k = 0.471 μN/m
- This matches the result from energy minimization to 3 significant figures



Need a Better Approximation?

- Add more terms to the polynomial
- Add other strain energy terms:
  - Shear: more significant as the beam gets shorter
  - Axial: more significant as deflections become larger
- Both of the above remedies make the math more complex, so encourage the use of math software, such as Mathematica, Matlab, or Maple
- Finite element analysis is really just energy minimization
- If this is the case, then why ever use energy minimization analytically (i.e., by hand)?
  - Analytical expressions, even approximate ones, give insight into parameter dependencies that FEA cannot
  - Can compare the importance of different terms
  - Should use in tandem with FEA for design