Lecture 15: Energy Methods

- Announcements:
- Module 9 (on "Energy Methods") and 10 (on "Resonance Frequency") are online
- HW#4 online, due Tuesday, 3/17, at 8 a.m.
- Midterm Exam: Remote Exam, Thursday, March 19, 9:30 a.m.-12:00 noon
- This is a video-recorded lecture, as will be subsequent lectures until the university goes back to ground classes
- We will go through the Midterm Info Sheet
- ------------------------------------
- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
- ------------------------------------
- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
- ------------------------------------
- Last Time:
- Working through energy methods
- Continue with this …

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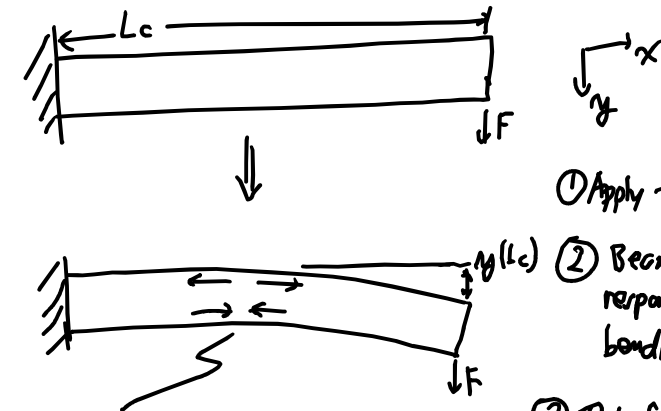
Same Problem as Before: Take a beam, apply a force.



① Apply force.

② Beam responds by bending.

③ This force has done work:

$W = F \cdot y(L_c)$

④ Strain generated.

So the beam has received an influx of stored energy.

magnitude of " " determined by shape.

⑤ Then

U: Stored Energy - Work Done → 0

When we choose the right shape.

This is how we get the beam's response to F!

transfer function

$y(x) = f(x)$

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Fundamentals: Energy Density

General Definition of Work:

$$\mathcal{W}(q_1) = \int_0^{q_1} e(q)\,dq$$

q: displacement
e: effort

for EE: $\mathcal{W}(Q) = \int_0^{Q} \frac{Q}{C}\,dQ$

Strain Energy Density

Value of strain @ position (x,y,z)

$$w = \int_0^{\epsilon_x} \sigma_x \, d\epsilon_x$$

$\sigma_x(\epsilon_x)$ → relates stress to strain @ position (x,y,z)

$[\sigma_x = E\epsilon_x]$

$$w = \int_0^{\epsilon_x} E\epsilon_x \, d\epsilon_x = \frac{1}{2}E\epsilon_x^2$$

Total Strain Energy: [J]

volume

$$\mathcal{W} = \iiint \left(\frac{1}{2}E\left(\epsilon_x^2+\epsilon_y^2+\epsilon_z^2\right) + \frac{1}{2}G\left(\gamma_{xy}^2+\gamma_{xz}^2+\gamma_{yz}^2\right)\right) dV$$

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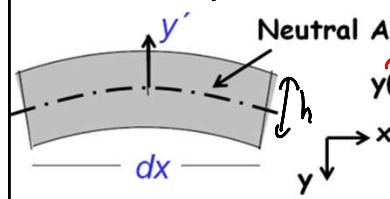
Bending Energy Density

W = width into the plane



same as z before

y(x) = transverse displacement of neutral axis

First, find the bending energy $d\mathcal{W}_{bend}$ in an infinitesimal length dx

$$d\mathcal{W}_{bend} = W\,dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\epsilon_x^2(y')\,dy'$$

$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \Rightarrow \epsilon_x(y') = y'\frac{d^2y}{dx^2}$$

$$d\mathcal{W}_{bend} = W\,dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\left[y'\frac{d^2y}{dx^2}\right]^2 dy'$$

$$= \frac{1}{2}E\underbrace{\left(\frac{Wh^3}{12}\right)}_{I_z}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{\mathcal{W}_{bend} = \frac{1}{2}EI_z\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}$$

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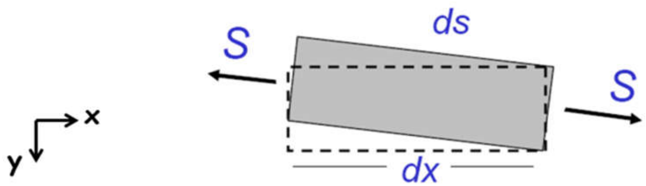
Energy Due to Axial Load



$\Rightarrow$ energy related lengthening:

$$ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

binomial theorem

$$\simeq dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore\ \epsilon_x = \frac{ds - dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

$$\left[dW_{axial} = S\epsilon_x dx = \frac{1}{2}S\left(\frac{dy}{dx}\right)^2 dx\right]$$

$$\boxed{W_{axial} = \frac{1}{2}S\int_0^L \left(\frac{dy}{dx}\right)^2 dx}$$

Axial Strain Energy

$\Rightarrow$ Look @ shear strain energy in your module.

- Go through Module 9, slides 10-18

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Estimating Resonance Frequency

$\mapsto x(t) = X_o \cos\omega_o t$

Potential Energy

$$W(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kX_o^2\cos^2\omega_o t$$

Kinetic Energy

$$K(t) = \frac{1}{2}m\dot{x}^2(t) = \frac{1}{2}mX_o^2\omega_o^2\sin^2\omega_o t$$

$\dot{x} = \frac{dx}{dt}$ = velocity

Remarks:

① Energy must be conserved.

② Total Energy = Potential Energy + Kinetic Energy at all times & locations on the structure

$$W_{max} = \frac{1}{2}kX_o^2 = K_{max} = \frac{1}{2}m\omega_o^2X_o^2$$

maximum potential energy

peak displacement

radian frequency

maximum kinetic energy

$*$

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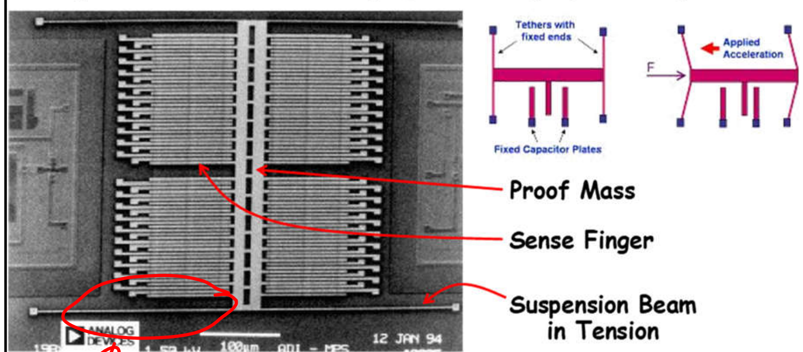
$\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$ ⇒ good for problems where mass & stiffness can be separated
i.e., they are distinct

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



In fabrication: purposely introduce a tensile stress in the beams!
a large one → why?
to avoid compression at all cost
buckling → dead device

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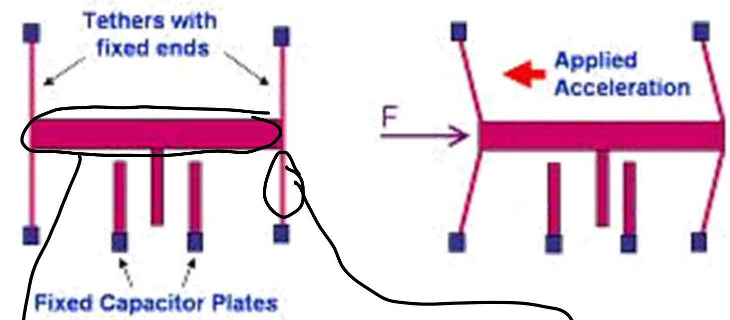

mass of structure >> mass of the springs
∴ ignore the mass of the springs

stiffness of the springs << stiffness of structure
∴ ignore the stiffness of the structure

For the ADXL-50, 60% the mass comes from the sense fingers → $M = 162$ ng

Suspension: Four tensioned beams

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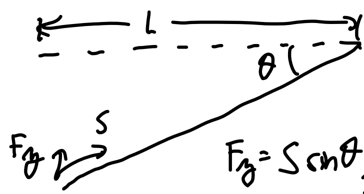
Bending Contribution

$$k_b = k_c \| k_c = \left(\frac{1}{k_c} + \frac{1}{k_c}\right)^{-1} = \frac{k_c}{2} = \frac{1}{2}\,\frac{3E(wh^3/12)}{(L/2)^3}$$

$$\Rightarrow k_b = Ew\left(\frac{h}{L}\right)^3 = 0.24\ N/m$$

Stretching Contribution



$$F_y = S\sin\theta \underset{\uparrow}{\simeq} S\left(\frac{y}{L}\right) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}} y$$

[assume small displacements]

k_{st} ↑ stretching stiffness

$$k_{st} = \frac{S}{L} = \frac{\sigma_r w h}{L} = 0.88\ N/m$$

Get the total spring constant

bending stiffness
stretching stiffness } parallel → add!

$$k_{tot} = 4(k_b + k_{st}) = 4(0.24 + 0.88) = \underline{4.5\ N/m}$$

Now, get the resonance freq:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.48\ N/m}{162\times10^{-12}\,kg}} = 26.5\ kHz$$

ADXL-50 Data Sheet: $f_0 = 24\,kHz$ ↔ difference?

↳ capacitive transducer → electrical stiffness

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