Lecture 16: Resonance Frequency

- **Announcements**:
- This is a video-recorded lecture, as will be subsequent lectures until the university goes back to ground classes
- No new homework (so you can enjoy your Spring Break) … but first …
- … Midterm Exam: Remote Exam, Thursday, March 19, 9:30 a.m.-12:00 noon
  - See Piazza post for procedural changes
  - Main differences:
    - No need to print out the exam
    - We will use Zoom proctoring
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- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
- ------------------------------------
- **Last Time**:
- Finished Lump Mass-Spring Approximation
- Now, address distributed mass & stiffness …

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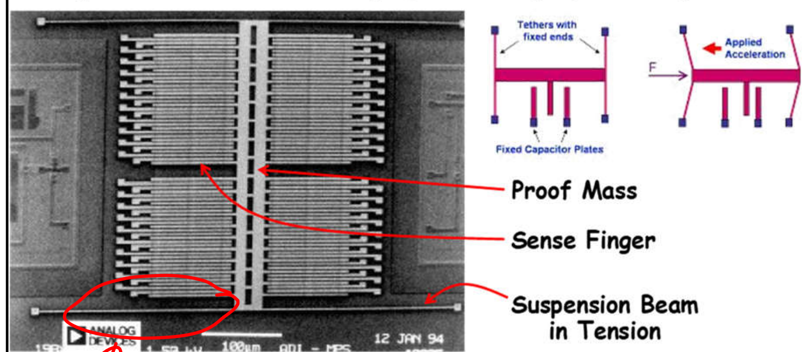
*: $\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$ ⇒ good for problems where mass & stiffness can be separated

i.e., they are distinct

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



In fabrication: purposely introduce a tensile stress in the beams!

a large one → why?

to avoid compression at all cost

buckling → dead device

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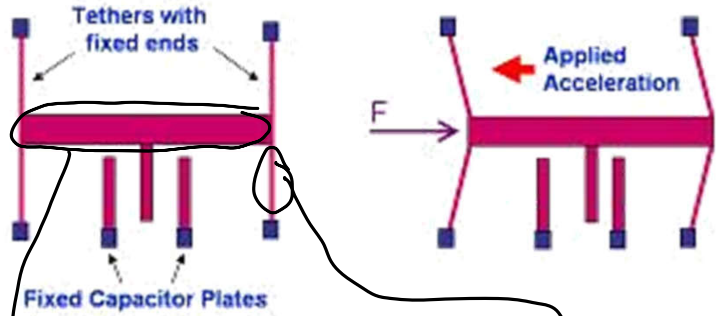


mass of structure >> mass of the springs

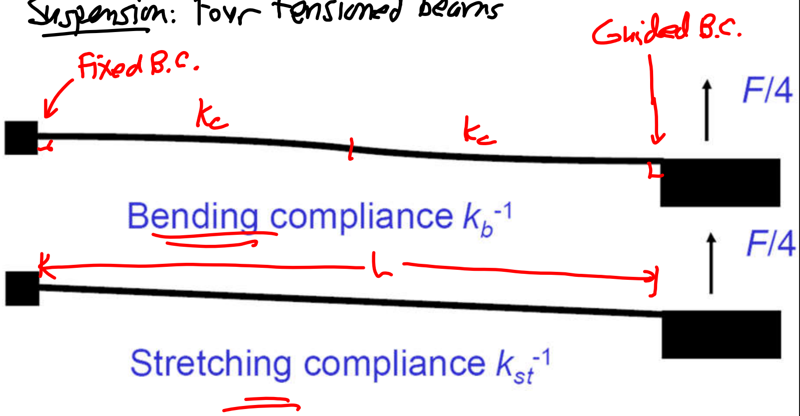
∴ ignore the mass of the springs

stiffness of the springs << stiffness of structure

∴ ignore the stiffness of the structure

For the ADXL-50, 60% the mass comes from the sense fingers ⟶ $M = 162\ ng$

Suspension: Four tensioned beams

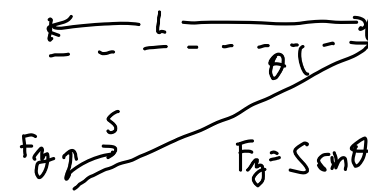


Bending Contribution

$$k_b = k_c \| k_c = \left(\frac{1}{k_c} + \frac{1}{k_c}\right)^{-1} = \frac{k_c}{2} = \frac{1}{2}\,\frac{3E(wh^3/12)}{(L/2)^3}$$

$$\Rightarrow k_b = Ew\left(\frac{h}{L}\right)^3 = 0.24\ N/m$$

Stretching Contribution



$$F_y = S\sin\theta \underset{\uparrow}{\simeq} S\left(\frac{y}{L}\right) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}} y$$

(assume small displacements)

k_{st} ↑ stretching stiffness

$$k_{st} = \frac{S}{L} = \frac{\sigma_r wh}{L} = 0.88\ N/m$$

Get the total spring constant

bending stiffness } parallel ⟶ add!
stretching stiffness }

$$k_{tot} = 4(k_b + k_{st}) = 4(0.24 + 0.88) = \underline{4.5\ N/m}$$

Now, get the resonance freq:

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.48\ N/m}{162\times10^{-12}\ kg}} = \boxed{26.5\ kHz}$$

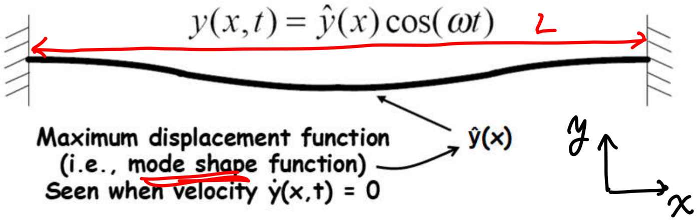
ADXL-50 Data Sheet: $f_o = 24\ kHz$ ⟷ difference?

⟶ capacitive transducer ⟶ electrical stiffness

Find the Resonance Frequency When Mass & Stiffness Are Distributed

- Vibrating structure displacement function:

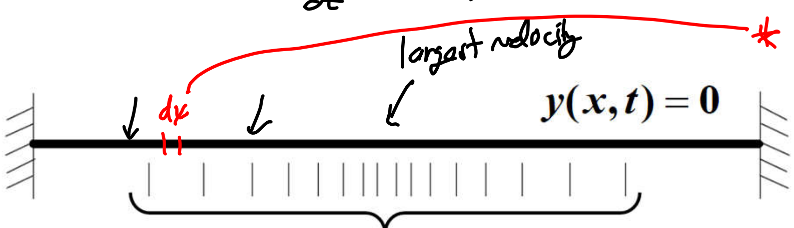
$$y(x,t) = \hat{y}(x)\cos(\omega t)$$



- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when $t=0$, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

Get Maximum Kinetic Energy

Velocity: $v(x,t) = \dfrac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin\omega t$



when $y(x,t)=0$, all the energy in the structure is kinetic ($W=0$) $\leftarrow t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \ldots$

$v\left(x, \frac{(2m+1)\pi}{2\omega}\right) = -\omega\hat{y}(x)$

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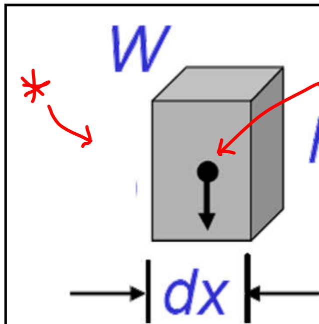

velocity: $v = -\omega\hat{y}(x)$ $\leftarrow \frac{(2m+1)\pi}{2\omega}$

$dK = \frac{1}{2}\cdot dm\cdot[v(x,t)]^2$

$dm = \rho(Wh\,dx)$ $\leftarrow$ density

Maximum Kinetic Energy, K_{max}:

$$K_{max} = \int_0^L \frac{1}{2}\rho Wh\,dx\,v^2(x,t) = \int_0^L \frac{1}{2}\rho Wh\omega^2[\hat{y}(x)]^2 dx$$

To get frequency: $K_{max} = W_{max}$

$$\therefore\ \omega = \sqrt{\frac{W_{max}}{\int_0^L \frac{1}{2}\rho Wh[\hat{y}(x)]^2 dx}} \quad [\text{radians/s}]$$

ω = radian resonance freq.
W_{max} = maximum potential energy
ρ = density of the structural material
W = beam width
h = " thickness
$\hat{y}(x)$ = resonance mode shape

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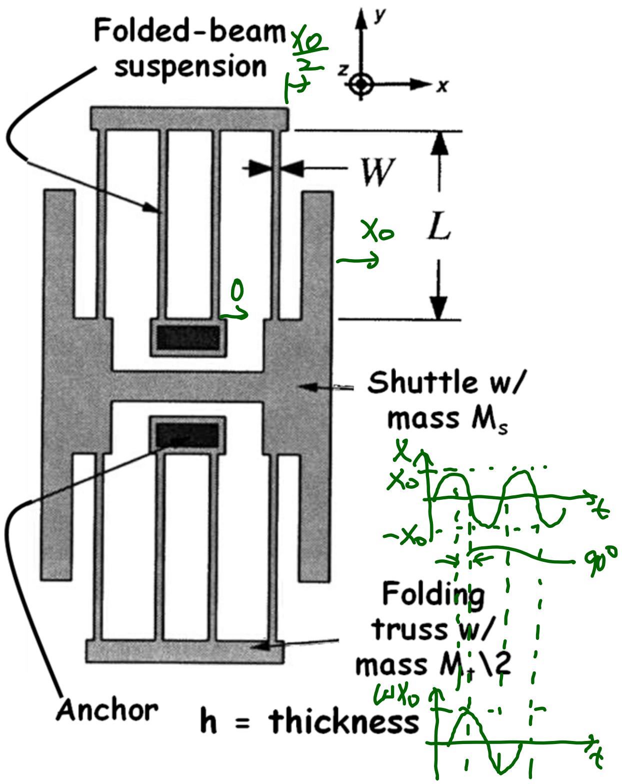
Resonance Freq. of a Folded Beam Structure



- Derive an expression for the resonance frequency of the above structure

Approximation)

⇒ m = shuttle mass
⇒ $k = k_c$

$$\omega_0 = \sqrt{\frac{k_c}{m}}$$

But not accurate enough for some applications.

⇒ for better accuracy, must integrate

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Use the Rayleigh-Ritz Method: (energy method)

$$KE_{max} = W_{max} \leftarrow \frac{1}{2}kx^2$$

Find the kinetic energy → one piece at a time

$$KE_{max} = \underbrace{K_s}_{\text{shuttle}} + \underbrace{K_t}_{\text{truss}} + \underbrace{K_b}_{\text{beams}}$$

$$= \frac{1}{2}v_s^2 M_s + \frac{1}{2}v_t^2 M_t + \frac{1}{2}\int v_b^2 \, dM_b$$

Velocity of the Shuttle: $v_s = \omega_0 X_0$

res. freq.; maximum displacement of shuttle

$$\therefore K_s = \frac{1}{2}v_s^2 M_s = \boxed{\frac{1}{2}\omega_0^2 X_0^2 M_s = K_s}$$

Velocity of Truss: $v_t = \frac{1}{2}v_s = \frac{1}{2}\omega_0 X_0$

$$\therefore K_t = \frac{1}{2}\left(\frac{1}{2}\omega_0 X_0\right)^2 M_t = \boxed{\frac{1}{8}\omega_0^2 X_0^2 M_t = K_t}$$

mass of both trusses

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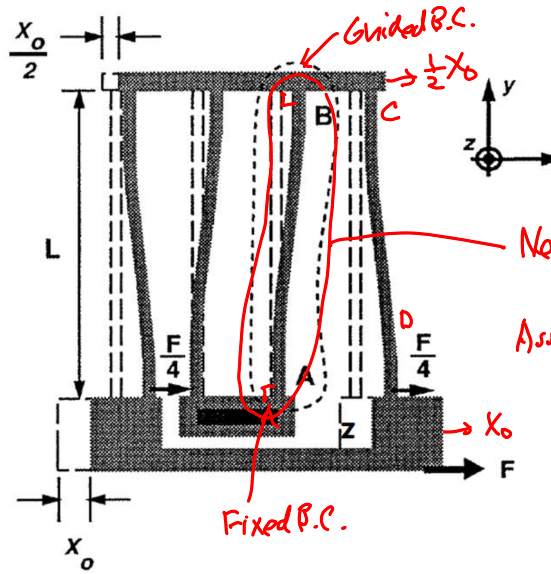
Velocity of the Beam Segments: → first beam [AB]



Need the mode shape.
⇓
Assume the mode shape is the same as the static displacement shape.

Segment [AB]:

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right), \quad 0 \le y \le L \qquad (1)$$

At $y=L$: $x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48EI_z}$ ← B.C.

Substitute into (1):

$$\hat{x}(y) = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\Big|_{[AB]} = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_0$$

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Plugging into expression for K_b:

$$K_{[AB]} = \frac{1}{2}\int_0^L \frac{X_0^2\omega_0^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$$= \frac{X_0^2\omega_0^2 M_{[AB]}}{8L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

mass per unit length

$M_{[AB]}$ = static mass

$$\boxed{K_{[AB]} = \frac{13}{280}X_0^2\omega_0^2 M_{[AB]}}$$

For segment [CD]:

$$v_b(y)\Big|_{[CD]} = X_0\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_0$$

Thus:

$$K_{[CD]} = \frac{X_0^2\omega_0^2 M_{[CD]}}{2L}\int_0^L \left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]^2 dy$$

$$\boxed{K_{[CD]} = \frac{83}{280}X_0^2\omega_0^2 M_{[CD]}}$$

static mass of beam [CD]

Let $M_b \triangleq$ total mass of all 8 beams

Thus:

$$K_b = 4K_{[AB]} + 4K_{[CD]} = \frac{6}{35}X_0^2\omega_0^2 M_b$$

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and

$$K_{max} = X_0^2 \omega_0^2 \left[\frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b \right]$$

← for the total mechanical ckt.

both trusses ↑ ↖ all beams

W_{max} → max. potential energy → equal to the work done to achieve maximum deflection

$$W_{max} = \frac{1}{2} k_x X_0^2$$

Then, using Rayleigh-Ritz:

$$K_{max} = W_{max}$$

$$\cancel{X_0^2} \omega_0^2 \left[\frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b \right] = \frac{1}{2} \cancel{k_x}^{\,k_c} \cancel{X_0^2}$$

$$\omega_0 = \left[\frac{k_c}{M_{eq}} \right]^{1/2}$$

where $M_{eq} = M_s + \frac{1}{4} M_t + \frac{12}{35} M_b$

(Resonance Freq. of a Folded-Beam Suspended Shuttle)

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