Lecture 18: Capacitive Transducers

- **Announcements**:
- Module 12 on Capacitive Transducers online
- HW#5 online; due Tuesday, 4/14
- Midterm solutions online w/ Lecture 17
- Project Definition online (discussed last time)
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- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - ↳ Energy Conserving Transducers
    - — Charge Control
    - — Voltage Control
  - ↳ Parallel-Plate Capacitive Transducers
    - — Linearizing Capacitive Actuators
    - — Electrical Stiffness
  - ↳ Electrostatic Comb-Drive
    - — 1st Order Analysis
    - — 2nd Order Analysis
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- **Last Time**:
- Finished our first pass on equivalent circuits
- Now, start on capacitive transducers …

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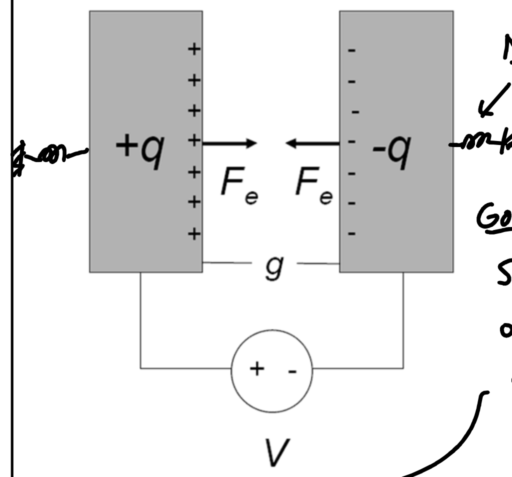
Basic Physics of Electrostatic Actuation



Note: Assume the plates are supported elastically.

Goal: Determine gap spacing g as a function of input variables q or V

1st: Determine the energy of the system.

2nd: Ask, What can I do to Δ the energy of the system?

① change the charge q

② change the separation g

$$\Delta W(q,g) = V\Delta q + F_e \Delta g$$

$$dW = V dq + F_e dg$$

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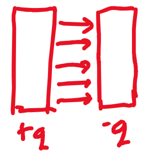
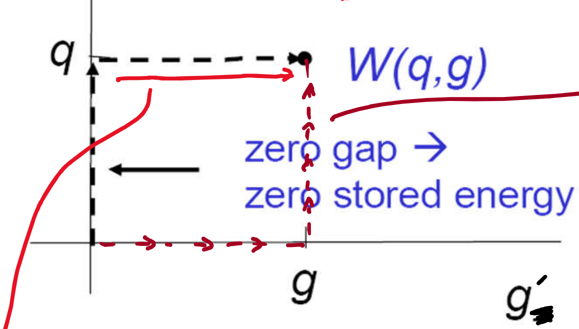
Stored Energy

$\mathcal{E} = \frac{q}{\epsilon A}$ ← plate overlap area





No change in charge: $dq = 0$

$dW = V\cancel{dq} + F_e dg$

$W = 0 + \int_0^g F_e \, dg'$

$F_e = \left(\frac{q}{2}\right)\mathcal{E} = \frac{1}{2}\frac{q^2}{\epsilon A}$ (independent of g)

$\therefore W = \int_0^g F_e \, dg' = F_e g' \Big|_0^g = F_e g$

$$\boxed{W(g) = \frac{1}{2}\frac{q^2}{\epsilon A} g}$$ **

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* Work done to charge a C to q at fixed gap g.

$dW = V dq + F_e \cancel{dg}^{\,0}$

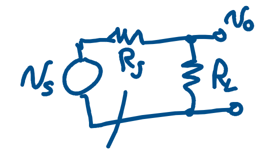
For a capacitor:

$q = CV \rightarrow V = \frac{q}{C}$

$\therefore W(q) = \int_0^q V dq' = \int_0^q \left(\frac{q'}{C}\right) dq = \frac{1}{2}\frac{q^2}{C}$

** $=$ $$\boxed{W(q) = \frac{1}{2}\frac{q^2}{\epsilon A} g}$$

V vs. I Drive:



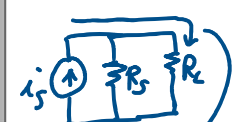
if R_s = small

$v_o = \left(\frac{R_L}{R_s + R_L}\right) v_s \cong v_s$

$\therefore$ voltage drive

if $R_s \gg R_L$, then

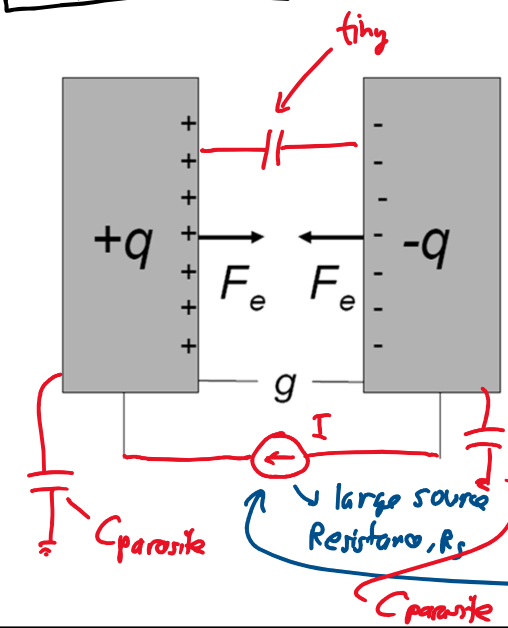
$v_o \ll v_s$



most i_s goes to R_L if $R_s \gg R_L$.

$\therefore$ Current Drive

Charge Control Case



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From $dW = V\,dq + F_e\,dg$

hold q = const. $\rightarrow V\,dq \rightarrow 0$

$dW = F_e\,dg \rightarrow F_e = \left.\frac{dW}{dg}\right|_{q=\text{const.}}$

$\Rightarrow$ Force is given by

$$F_e = \left.\frac{\partial W(q,g)}{\partial g}\right|_{q=\text{const.}} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A}g\right)$$

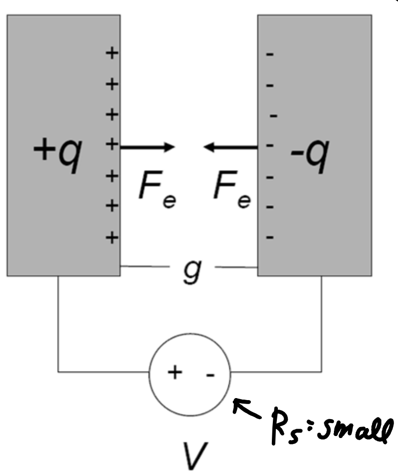
$$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}} \Rightarrow \text{indep. of gap spacing!}$$

$\Rightarrow$ voltage is given by:

$$V = \left.\frac{\partial W(q,g)}{\partial q}\right|_{g=\text{const.}} = \frac{\partial}{\partial q}\left(\frac{1}{2}\frac{q^2}{\epsilon A}g\right) = \frac{qg}{\epsilon A} \rightarrow \boxed{V = \frac{q}{C}} \checkmark$$

(consistent w/ what we already know)

Voltage Control



Want to write $F_e = f(V)$

We know this: $dW = V\,dq + F_e\,dg$

$W = W(q,g)$

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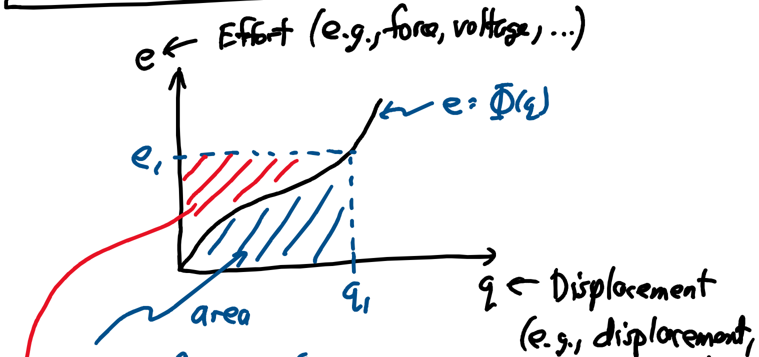
Need: $W'(V,g)$

↑ need to replace charge q w/ voltage V

Can get this using a Legendre transformation.

Energy & Co-Energy



$$W(q_1) = \int_0^{q_1} e\,dq = \int_0^{q_1} \Phi(q)\,dq$$

Co-Energy:

$$W'(e_1) = \int_0^{e_1} q\,de = \int_0^{e_1} \Phi^{-1}(e)\,de$$

For a linear system, these will be equal.

Can define co-energy as:

$$\underset{\text{co-energy}}{W'(e)} = eq - \underset{\text{energy}}{W(q)} \quad \text{(from the plot)} \quad *$$

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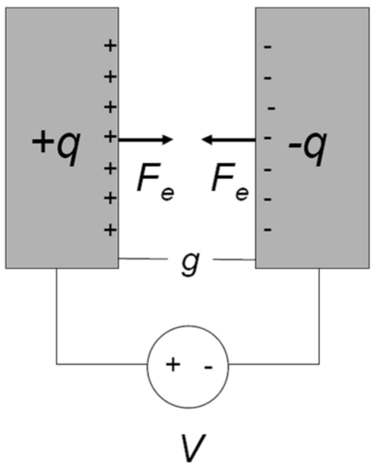
Co-Energy Formulation for Voltage-Control



* $W'(V,g) = Vq - W(q,g)$

Differentially, this becomes

$$dW'(V,g) = (q\,dV + \cancel{V\,dq}) - dW(q,g)$$

$$[dW(q,g) = F_e\,dg + \cancel{V\,dq}]$$

$$\boxed{dW'(V,g) = q\,dV - F_e\,dg}$$ ← Working Co-Energy Expression

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Find co-energy in terms of voltage, V:

$$W' = \int_0^V q(g,V')\,dV' = \int_0^V \left(\frac{\epsilon A}{g}\right) V'\,dV'$$

$$= \frac{1}{2}\left(\frac{\epsilon A}{g}\right) V^2 = \frac{1}{2} C V^2 \checkmark \quad \text{(as expected)}$$

Electrostatic (a Voltage-Controlled) Force:

$$F_e = -\left.\frac{\partial W'(V,g)}{\partial g}\right|_{V=\text{const.}}$$

$$= +\frac{1}{2}\left(\frac{\epsilon A}{g^2}\right) V^2 = \boxed{\frac{1}{2}\frac{C}{g} V^2 = F_e}$$

depends on gap!

Charge:

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_{g=\text{const.}} = \frac{\epsilon A}{g} V = CV \checkmark$$

(as expected)

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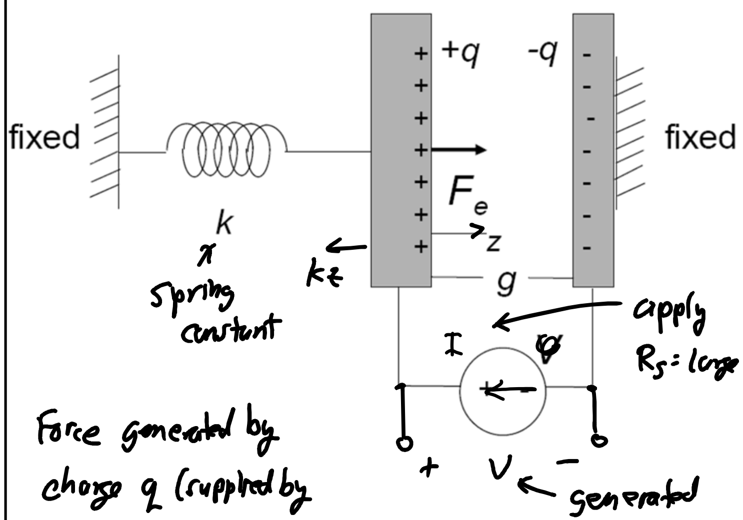
Charge-Control of a Spring-Suspended C



Force generated by charge q (supplied by current I):

$$F_e = \frac{\partial W(q,g)}{\partial g}\Big|_q = \frac{q^2}{2\epsilon A}$$

Restoring force of the spring: $F_{spring} = kz = F_e$

equilibrium

The gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

initial gap

q↑ can drive g→0 in a continuous fashion

$$V = \frac{q}{C} = \frac{q}{\epsilon A}g = \boxed{\frac{q}{\epsilon A}\left(g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V} \rightarrow V\downarrow \text{ as } g\downarrow$$

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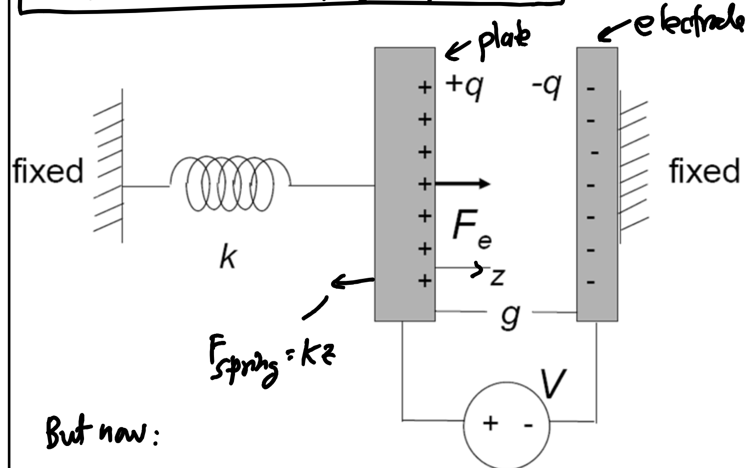
Voltage-Control of a Spring-Suspended C



But now:

$$F_e = -\frac{\partial W'(V,g)}{\partial g}\Big|_g \Rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{\epsilon A}{g^2}V^2 = g}$$

initial gap spacing

g show up on both sides!

If V↑ → g↓ → Fe↑

(+) Feedback!

If loop gain > 1, then this will go unstable!

plate will collapse into the electrode!

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Charge: (for a stable gap)

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_g = CV \checkmark \quad \text{(as expected)}$$

$\boxed{\text{Stability Analysis}}$

⇒ determine under what conditions voltage-control will cause collapse of the plates:

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_0 - g)}_{F_{spring}}$$

What happens when we change g by a small increment dg?
↳ get an increment in the net attractive force F_{net}

$$\underset{(-)}{\underline{dF_{net}}} = \frac{\partial F_{net}}{\partial g} dg = \left[-\frac{\epsilon A V^2}{g^3} + k \right] \underset{(-)}{\underline{dg}}$$

If $g\downarrow \rightarrow dg = (-)$, then for stability, need $F_{net}\downarrow \rightarrow dF_{net} = (-)$

This must be (+)! → otherwise, the plates collapse

Thus: $\boxed{k > \frac{\epsilon A V^2}{g^3}}$ (for a stable uncollapsed system)

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$\boxed{\text{Pull-in Voltage \& Pull-in Gap}}$

$V_{PI} \triangleq$ voltage @ which plates collapse

$g_{PI} \triangleq$ gap @ " " "

The plate goes unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \quad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_0 - g_{PI}) \quad (2)$$

Insert (1) into (2):

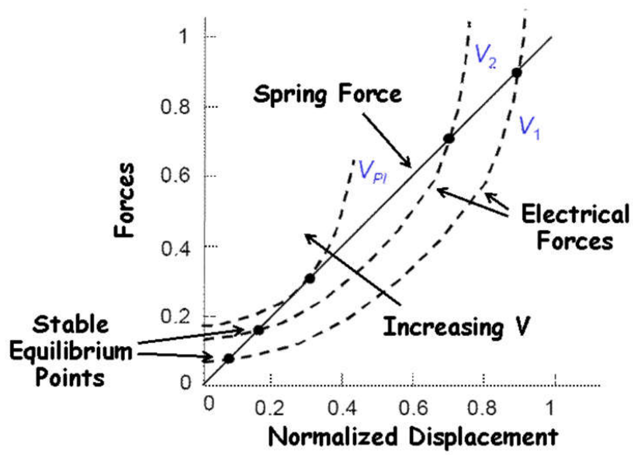
$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2\cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_0 - g_{PI})$$

$$\frac{g_0 - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_0 = \frac{3}{2} g_{PI} \rightarrow \boxed{g_{PI} = \frac{2}{3} g_0}$$

When the gap is driven by a voltage to (2/3) the initial gap → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow \boxed{V_{PI} = \sqrt{\frac{8}{27}\frac{k g_0^3}{\epsilon A}}}$$

← stay under this to prevent collapse



Advantages of Electrostatic Actuators:

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed $\rightarrow$ low cost!
- Energy conserving $\rightarrow$ only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink $\rightarrow$ electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

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