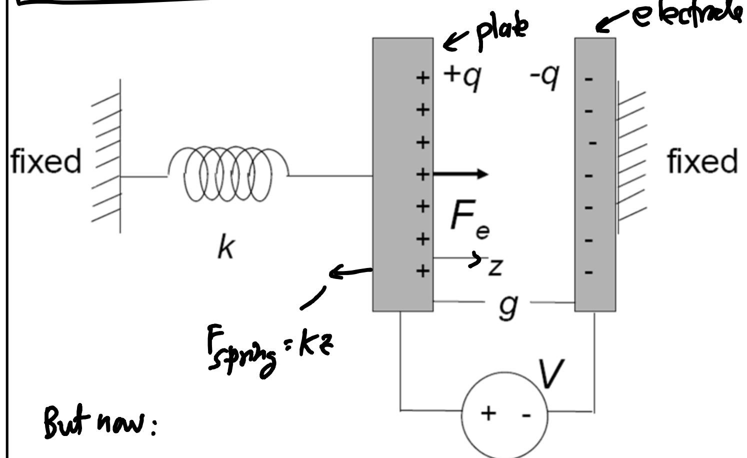
Lecture 19: Electrical Stiffness

- Announcements:
- Project Slide Set #1 due Friday, April 10
- HW#5 online and due Tuesday, 4/14, at 8 a.m.
- I sent everyone Z scores
- ------------------------------------
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- ------------------------------------
- Last Time:
- Finished pull in
- Now, continue with this …

1

Voltage-Control of a Spring-Suspended C



But now:

$$F_e = -\frac{\partial W'(V,g)}{\partial g}\Big|_q \Rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{\epsilon A}{g^2}V^2 = g}$$

initial gap spacing

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1, then this will go unstable!

plate will collapse into the electrode!

2

Charge: (for a stable gap)

$$q = \frac{\partial W'(V,g)}{\partial V}\bigg|_g = CV \checkmark \quad \text{(as expected)}$$

Stability Analysis

⇒ determine under what conditions ~~voltage-control~~ will cause collapse of the plates:

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_o - g)}_{F_{spring}}$$

What happens when we change g by a small increment dg?

↳ get an increment in the net attractive force F_{net}

$$\underset{(-)}{dF_{net}} = \frac{\partial F_{net}}{\partial g} dg = \left[-\frac{\epsilon A V^2}{g^3} + k\right] \underset{(-)}{dg}$$

If $g\downarrow \to dg = (-)$, then for stability, need $F_{net}\downarrow \to dF_{net} = (-)$

This must be (+)! → otherwise, the plates collapse

Thus: $\boxed{k > \frac{\epsilon A V^2}{g^3}}$ (for a stable uncollapsed system)

3

Pull-in Voltage & Pull-in Gap

$V_{PI} \triangleq$ voltage @ which plates collapse

$g_{PI} \triangleq$ gap @ " " "

The plate goes unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \quad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_o - g_{PI}) \quad (2)$$

Insert (1) into (2):

$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2\cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_o - g_{PI})$$

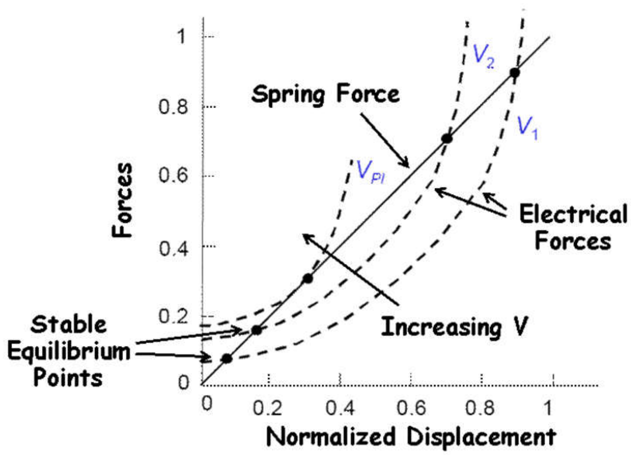
$$\frac{g_o - g_{PI}}{g_{PI}} = \frac{1}{2} \to g_o = \frac{3}{2} g_{PI} \to \boxed{g_{PI} = \frac{2}{3} g_o}$$

When the gap is driven by a voltage to (2/3) the initial gap → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \to \boxed{V_{PI} = \sqrt{\frac{8}{27}\frac{k g_o^3}{\epsilon A}}}$$

← stay under this to prevent collapse

4



Advantages of Electrostatic Actuators:

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Capacitive transducers are generally much stronger than piezoelectric at low frequencies, e.g., for many sensor applications
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

5

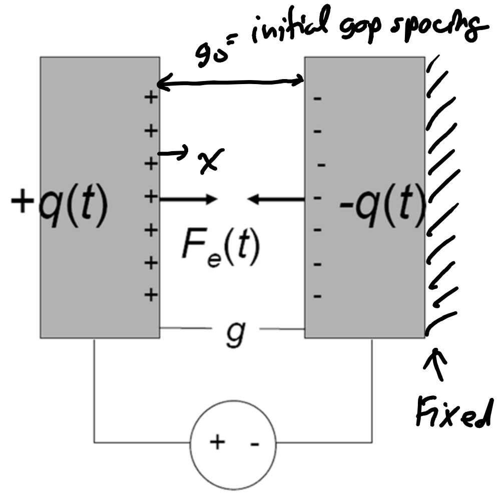
Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- At high frequencies, relatively weak compared with other transducers (e.g., piezoelectric)
  - Due to higher mechanical stiffness and smaller electrode overlap area
  - but things get better as dimensions scale with a fourth power dependence on gap spacing

- *Go through variable naming convention in slide 21 of Lecture Module 12*

6

Linearizing the Voltage-to-Force Transfer Function



$$v(t) = \underbrace{V_P}_{\substack{\text{DC Bias} \\ \text{(large voltage)}}} + \underbrace{v_i(t)}_{\substack{\text{signal (AC)} \\ \text{(generally small)}}}$$

$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}C[v(t)]^2\right]$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}[v(t)]^2 = \frac{1}{2}\frac{\partial C}{\partial x}[V_P + v_i(t)]^2$$

$$= \frac{1}{2}\left(V_P^2 + 2V_P v_i(t) + \cancel{[v_i(t)]^2}\right)\frac{\partial C}{\partial x}$$

$$\boxed{V_P \gg v_i(t)} \Rightarrow F_e(t) = \underbrace{\frac{1}{2}V_P^2\frac{\partial C}{\partial x}}_{\text{DC Offset}} + \underbrace{V_P\frac{\partial C}{\partial x}v_i(t)}_{\text{AC Drive Signal}} \text{ --- } *$$

$$C_0 = \frac{\epsilon A}{g_0} \rightarrow C(x) = \frac{\epsilon A}{g_0 - x} = C_0\left(1 - \frac{x}{g_0}\right)^{-1}$$

$$\boxed{x \ll g_0} \Rightarrow \approx C_0\left(1 + \frac{x}{g_0}\right)$$

$$\frac{\partial C}{\partial x} = \frac{C_0}{g_0} = \frac{\epsilon A}{g_0^2}$$

V_P provides "gain"

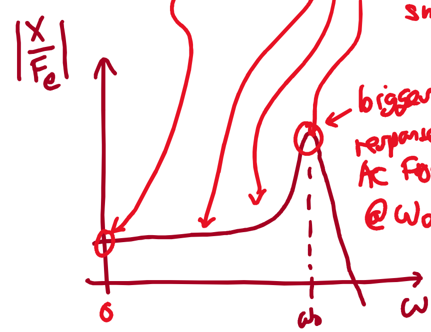
$$* \rightarrow F_e(t) = \underbrace{\frac{1}{2}\frac{C_0}{g_0}V_P^2}_{\text{DC Offset}} + \underbrace{V_P\frac{C_0}{g_0}v_i(t)}_{\text{AC term}}$$

~ constant for small amplitude

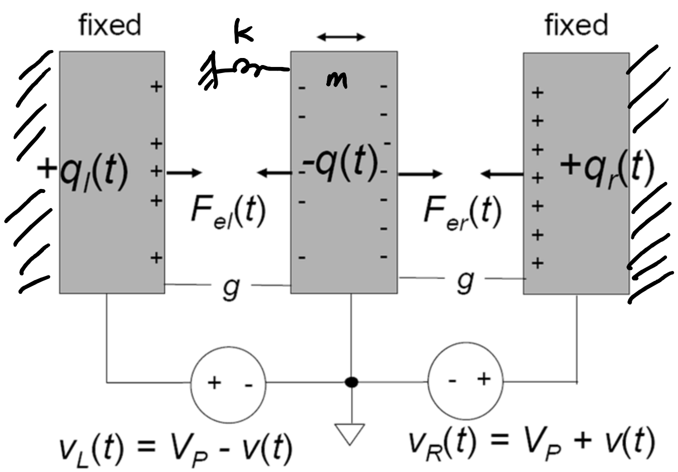
$\therefore$ this has mostly a linear dependence!

linearized in the small-signal sense

only holds for small amplitudes when $x \ll g_0$



Can Cancel the DC Offset Via Differential Symmetry



$$F_{net}(t) = F_{er}(t) - F_{el}(t)$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{[v_R(t)]^2 - [v_L(t)]^2\right\}$$

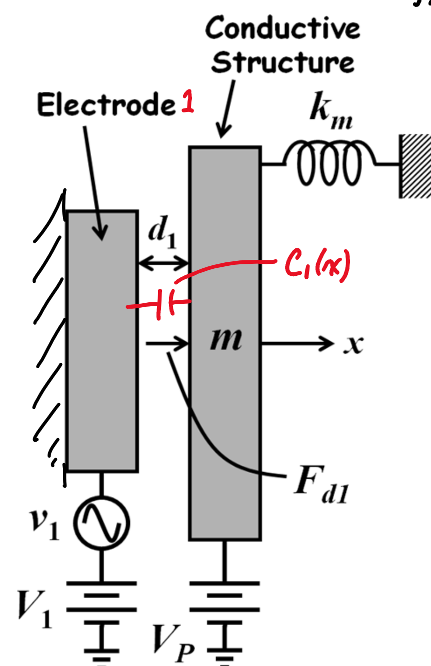
$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{\cancel{V_P^2} + 2V_P v(t) + \cancel{[v(t)]^2} - \left(\cancel{V_P^2} - 2V_P v(t) + \cancel{[v(t)]^2}\right)\right\}$$

$$\boxed{F_{net}(t) = 2V_P\frac{\partial C}{\partial x}v(t) = \underbrace{2V_P\frac{C_o}{g_o}}_{\text{gain term}}v(t)} \leftarrow \text{Fairly linear w/ } v(t)!$$

⇒ no DC offset, but still can pull in due to non-idealities!

Nonlinearity Still Effects Us

(even when we use small-signals & balanced differential electrodes)



More Complete Expressions

$$C_1(x) = \frac{\epsilon A}{d_1 + x} = C_{o1}\left(1 + \frac{x}{d_1}\right)^{-1} \rightarrow \frac{\partial C_1}{\partial x} = -\frac{C_{o1}}{d_1}\left(1 + \frac{x}{d_1}\right)^{-2}$$

[Expand into Taylor series]

$$\frac{\partial C_1}{\partial x} = -\frac{C_{o1}}{d_1}\left(1 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots\right)$$

where $A_1 = -\frac{2}{d_1}$, $A_2 = \frac{3}{d_1^2}$, $A_3 = -\frac{4}{d_1^3}$, ...

$$F_{d1} = \frac{1}{2}\frac{\partial C_1}{\partial x}(V_P - V_1 - v_1)^2 = \frac{1}{2}\frac{\partial C}{\partial x}(V_{P1} - v_1)^2$$

$V_{P1} = V_P - V_1$

[small displacements: $x \ll d_1$]

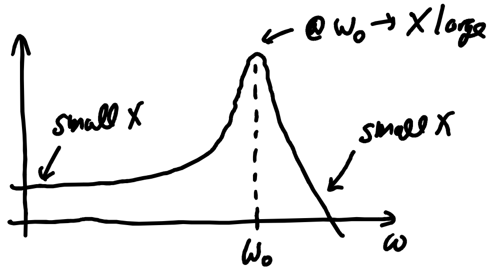
$\frac{1}{2}(1+\cos 2\omega_0 t)$ not @ resonance

$\cos^2\omega_0 t$

$$F_{d1} = \frac{1}{2}\left(-\frac{C_{01}}{d_1}\right)(1+A_1 x)(V_{P1}^2 - 2V_{P1}v_1 + v_1^2)$$

$$= \frac{1}{2}\left(-\frac{C_{01}}{d_1}\right)\left\{V_{P1}^2 - 2V_{P1}v_1 + v_1^2 + A_1 V_{P1}^2 x - 2A_1 V_{P1} x v_1 + \cancel{A_1 x v_1^2}\right\}$$

Resonance: $\left|\frac{X}{F_{d1}}\right|$



@ resonance:

$$x = \frac{QF_{d1}}{jk} = \frac{Q}{jk}\frac{\partial C_1}{\partial x}V_{P1}v_1$$

90° phase shift

$$v_1 = |v_1|\cos\omega_0 t \longrightarrow x = |x|\sin\omega_0 t$$

90° phase-shifted

Force terms @ ω_0

k_e → electrical stiffness

$$F_{d1}\big|_{\omega_0} = V_{P1}\frac{C_{01}}{d_1}|v_1|\cos\omega_0 t + V_{P1}^2\frac{C_{01}}{d_1^2}|x|\sin\omega_0 t$$

drive force term

proportional to x

90° phase-shifted f/

∴ in phase w/ displacement!

∴ it's a stiffness

Electrical Stiffness:

① A negative spring constant!

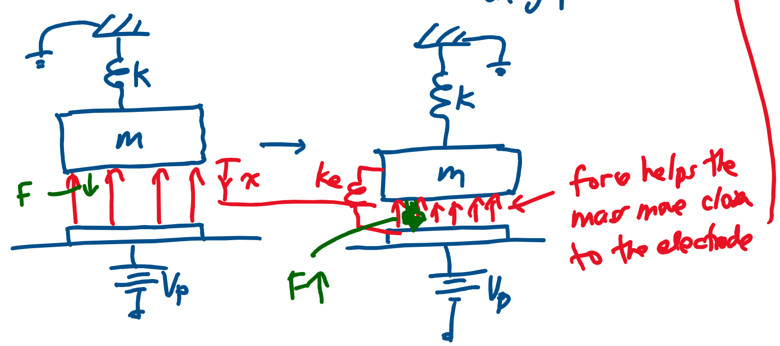
② Derives from V_P:

$$\boxed{k_e = V_{P1}^2\frac{C_{01}}{d_1^2} = V_{P1}^2\frac{\epsilon A}{d_1^3}}$$

overlap area of C

DC Bias

3rd power dependence on gap



$k_e \rightarrow$ can affect resonance frequency

$\omega_0 \triangleq$ radian resonance freq. w/ no V_P applied (i.e., $V_P = 0$)

(nominal resonance freq.)

$\omega_0 = \sqrt{\frac{k_m}{m}} \leftarrow$ mechanical stiffness

$$\omega_0' = \sqrt{\frac{k_{tot}}{m}} = \sqrt{\frac{k_m - k_e}{m}} = \underbrace{\sqrt{\frac{k_m}{m}}}_{\omega_0}\left(1 - \frac{k_e}{k_m}\right)^{1/2}$$

$$\therefore \boxed{\omega_0' = \omega_0\left[1 - \frac{V_{P1}^2}{k_m}\frac{\epsilon A}{d_1^3}\right]^{1/2}}$$

now a function of DC Bias!

(voltage-controllable!)

13