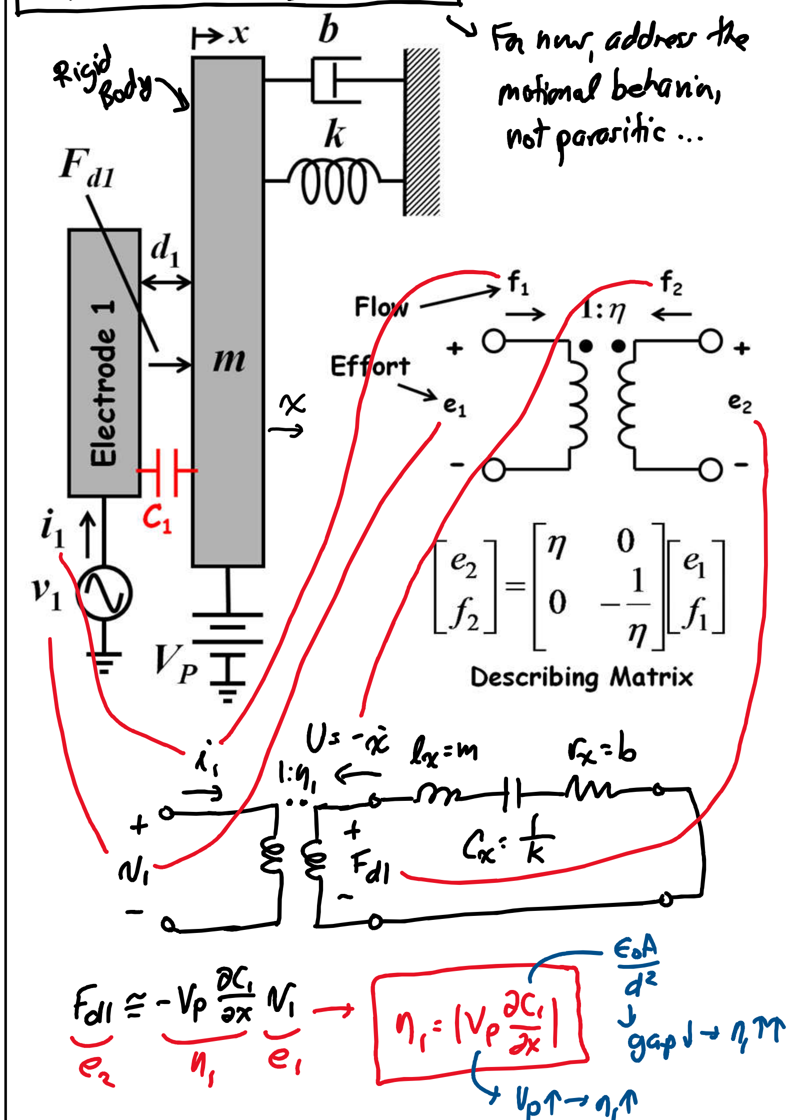
Lecture 21: Equivalent Circuits II

- **Announcements:**
- Project Slide Set #2 due Friday, April 17
- HW#6 online soon
- Module 13 on "Equivalent Circuits II" online
- Module 15 on "Gyros, Noise, & MDS" online
- -----------------------------------
- Reading: Senturia, Chpt. 6, Chpt. 14
- Lecture Topics:
  - Input Modeling
    - —Force-to-Velocity Equiv. Ckt.
    - —Input Equivalent Ckt.
  - Current Modeling
    - —Output Current Into Ground
    - —Input Current
    - —Complete Electrical-Port Equiv. Ckt.
  - Impedance & Transfer Functions
- -----------------------------------
- Reading: Senturia, Chpt. 14, Chpt. 16, Chpt. 21
- Lecture Topics:
  - Gyroscopes
- Reading: Senturia, Chpt. 14
- Lecture Topics:
  - Detection Circuits
    - —Velocity Sensing
    - —Position Sensing
- -----------------------------------
- Last Time: Output current into ground
- Now, continue towards a complete equivalent circuit

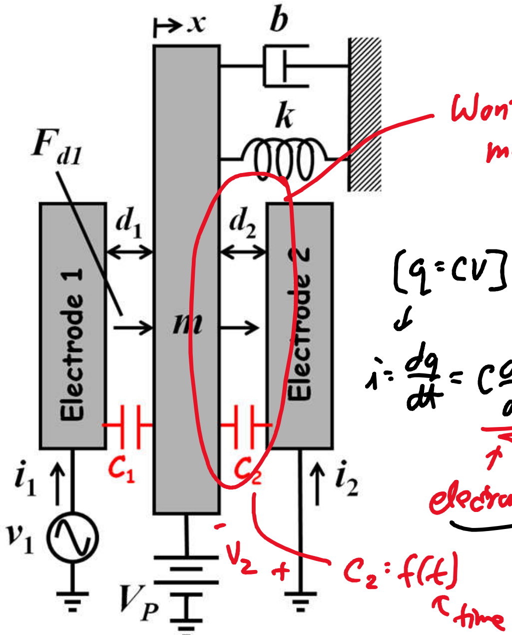
1

Input Electrical Equivalent Ckt.



2

Output Current Into Ground



Want this ckt. model.

$[q = CV]$

$$i = \frac{dq}{dt} = \underbrace{C\frac{dV}{dt}}_{\text{electronics}} + \underbrace{V\frac{dC}{dt}}_{\text{MEMS}}$$

$$i_2 = C_2(x,t)\cancelto{0}{\frac{dV_2(t)}{dt}} + V_2(t)\frac{dC_2(x,t)}{dt}$$

$$[V_2(t) = -V_P] \Rightarrow i_2 = -V_P\frac{dC_2}{dt} = -V_P\frac{\partial C_2}{\partial x}\frac{\partial x}{\partial t}$$

In phasor form: $I_2(j\omega) = -V_P\frac{\partial C_2}{\partial x}(j\omega X)$

$$\boxed{I_2(j\omega) = -j\omega V_P\frac{\partial C_2}{\partial x}X}$$

motional current

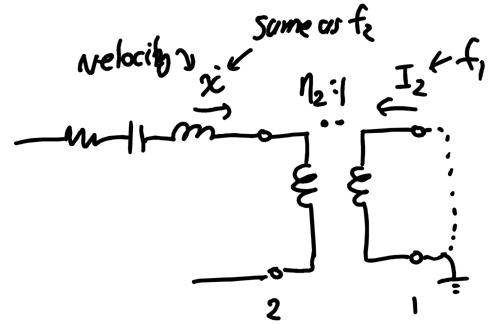
3

$$I_2(j\omega) = -j\omega V_P\frac{\partial C_2}{\partial x}X = -V_P\frac{\partial C_2}{\partial x}\dot{x}$$

velocity

90° phase lag (+) (+) → $I_2 = (-)$ when $x = (+)$ ✓

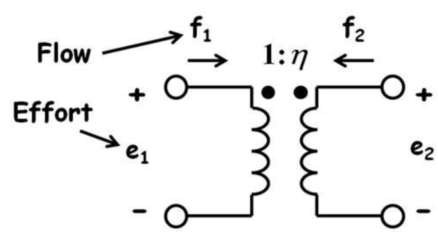
=



$$f_2 = -\frac{1}{\eta_2}f_1 \rightarrow f_1 = -\eta_2 f_2$$

$$[f_1 = I_2,\ f_2 = \dot{x}] \Rightarrow I_2 = -\eta_2\dot{x}$$

$$\therefore \boxed{\eta_2 = \left|V_P\frac{\partial C_2}{\partial x}\right|}$$

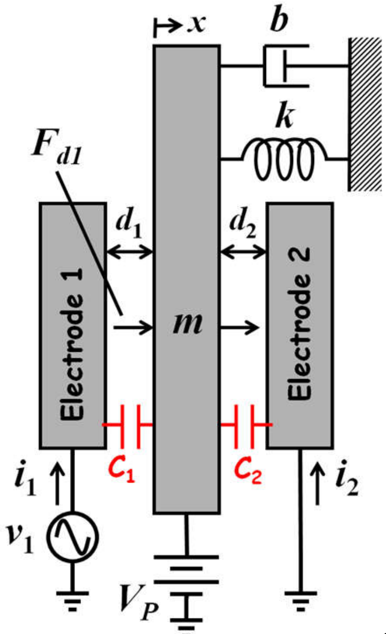


$$\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\frac{1}{\eta} \end{bmatrix}\begin{bmatrix} e_1 \\ f_1 \end{bmatrix}$$

Describing Matrix

4

Input Current Expression



Get $I_1(j\omega)$: cap, lower case } $V_1(t)$ is a total voltage

$$i_1(t) = C_1(x,t)\frac{dV_1(t)}{dt} + V_1(t)\frac{dC_1(x,t)}{dt}$$

$[V_1(t) = v_1 - V_P]$ → $i_1 = C_1\frac{dv_1}{dt} + [v_1 - V_P]\frac{\partial C_1}{\partial x}\frac{\partial x}{\partial t}$

f(t)

$$\therefore I_1(j\omega) = \underbrace{j\omega C_1 V_1}_{\text{Feedthrough Current}} + j\omega V_1 \cancel{\frac{\partial C_1}{\partial x}X} - \underbrace{j\omega V_P\frac{\partial C_1}{\partial x}X}_{\text{Motional Current}}$$

due to motion!

@DC: $x = \frac{F_{d1}}{k} = -\frac{1}{k}V_P\left(\frac{\partial C_1}{\partial x}\right)V_1$

@resonance: $x = \frac{QF_{d1}}{jk} = -\frac{Q}{jk}V_P\frac{\partial C_1}{\partial x}V_1 = X$

90° lag

Thus: (@ resonance) → ω_o

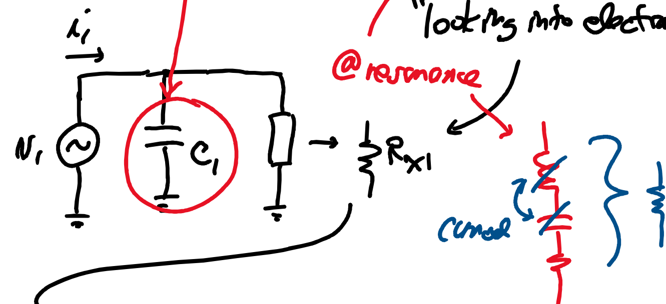
$$I_1(j\omega_o) = j\omega_o C_1 V_1 + j\omega_o\left(V_P\frac{\partial C_1}{\partial x}\right)^2\frac{Q}{jk}V_1$$

$$= j\omega_o C_1 V_1 + \omega_o\frac{Q}{k}\eta_{e1}^2 V_1$$

90° phase-shifted from v_1 → This is a capacitor in shunt w/ the input!

In phase w/ v_1 → This is an effective resistance seen "looking into electrode 1"

@resonance

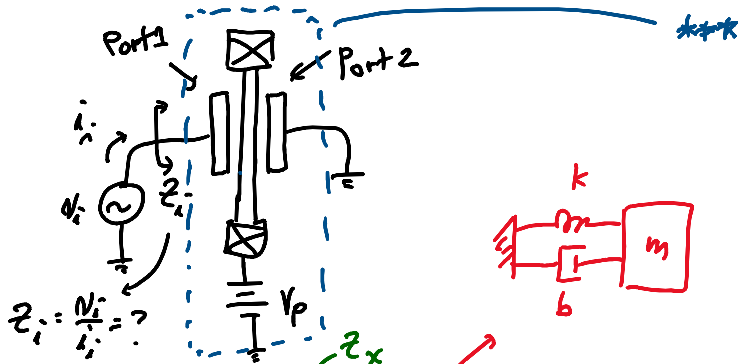


Motional Resistance:

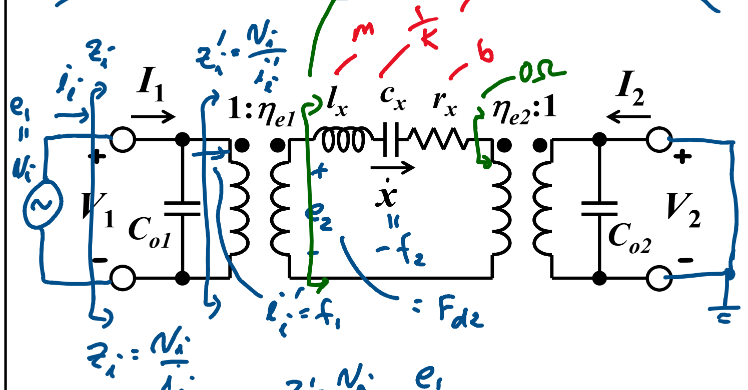
$$R_{x1} = \frac{V_1}{I_1} = \frac{k}{\omega_o Q\eta_{e1}^2} = \frac{m\omega_o}{Q\eta_{e1}^2} = \boxed{\frac{b}{\eta_{e1}^2} = R_{x1}}$$ ✓ → **

"motion"

Input Impedance Into Port 1



$Z_i = \frac{v_i}{i_i} = ?$



$Z_i = \frac{v_i}{i_i}$

$Z_i' = \frac{v_i}{i_i'} = \frac{e_1}{f_1}$

$$\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\frac{1}{\eta} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \Rightarrow \begin{array}{l} e_2 = \eta e_1 \rightarrow e_1 = \frac{e_2}{\eta} \\ f_2 = -\frac{1}{\eta} f_1 \rightarrow f_1 = -\eta f_2 \end{array}$$

$$\frac{e_1}{f_1} = \frac{e_2}{\eta}\left(\frac{1}{-\eta f_2}\right) = -\frac{1}{\eta^2}\frac{e_2}{f_2} \rightarrow \frac{v_i}{i_i'} = Z_i' = -\frac{1}{\eta_{e1}^2}\frac{F_{d2}}{(-\dot{x}_2)} = \rightarrow *$$

7

$$* \rightarrow \frac{v_i}{i_i'} = -\frac{1}{\eta_{e1}^2}\frac{F_{d2}}{(-\dot{x}_2)} = \frac{1}{\eta_{e1}^2} Z_x$$

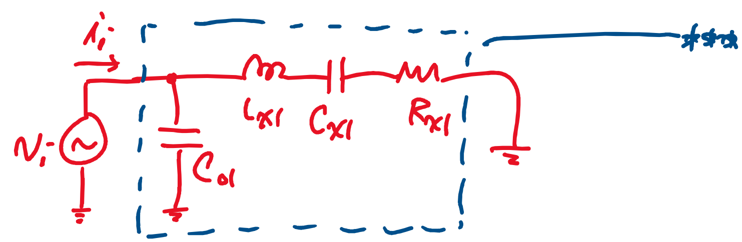
$$Z_x = j\omega l_x + \frac{1}{j\omega c_x} + r_x$$

$$Z_i' = \frac{1}{\eta_{e1}^2}\left(j\omega l_x + \frac{1}{j\omega c_x} + r_x\right)$$

$$= \underbrace{j\omega\left(\frac{l_x}{\eta_{e1}^2}\right)}_{L_{x1}} + \underbrace{\frac{1}{j\omega(\eta_{e1}^2 c_x)}}_{C_{x1}} + \underbrace{\frac{r_x}{\eta_{e1}^2}}_{R_{x1}}$$

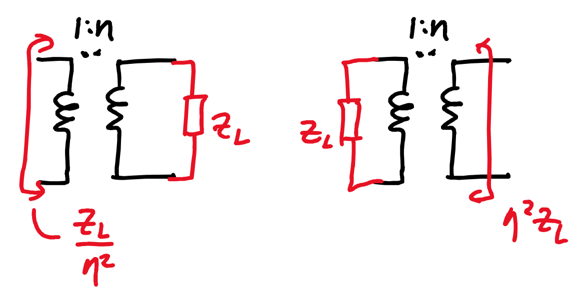
$r_x \leftarrow b$

$R_{x1} = \frac{b}{\eta_{e1}^2}$ ✓ — **



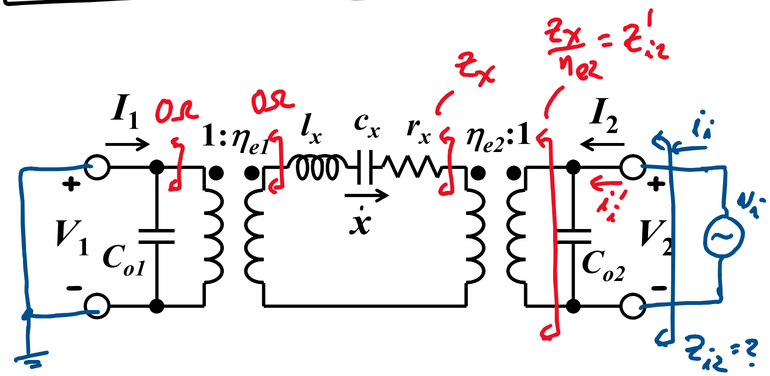
Purely Electrical Equivalent Ckt.

Xformer Inspection Analysis

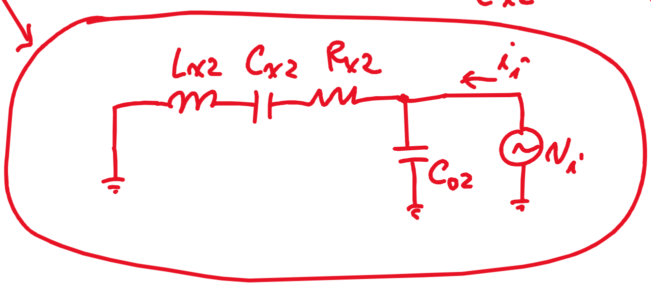


8

Input Impedance Into Port 2



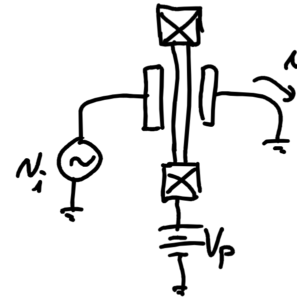
$$Z'_{i2} = \frac{v_i}{i'_i} = \frac{Z_x}{\eta_{e2}^2} = j\omega\underbrace{\left(\frac{l_x}{\eta_{e2}^2}\right)}_{L_{x2}} + \underbrace{\frac{1}{j\omega(\eta_{e2}^2 c_x)}}_{C_{x2}} + \underbrace{\frac{r_x}{\eta_{e2}^2}}_{R_{x2}}$$

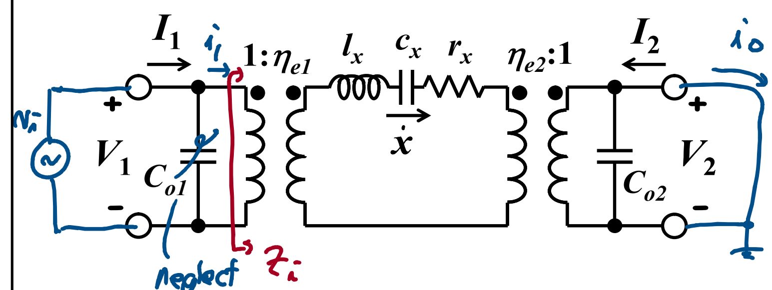


9

Port 1 to Port 2 Transconductance

$$G = \frac{i_o}{v_i}$$





$\dot{x} = \frac{1}{\eta_{e1}} i_1$

$i_o = \eta_{e2}\dot{x}$

$$i_o = \frac{\eta_{e2}}{\eta_{e1}} i_1 = \frac{\eta_{e2}}{\eta_{e1}}\left(\frac{v_i}{Z_i}\right) \quad \text{(neglecting } C_o\text{)}$$

$$= \frac{\eta_{e2}}{\eta_{e1}} v_i \left[\eta_{e1}^2 \frac{1}{j\omega l_x + \frac{1}{j\omega c_x} + r_x}\right]$$

$$\therefore \frac{i_o}{v_i}(j\omega) = \frac{\eta_{e1}\eta_{e2}}{j\omega l_x + \frac{1}{j\omega c_x} + r_x}$$

$$= \left[j\omega L_{x12} + \frac{1}{j\omega C_{x12}} + R_{x12}\right]^{-1}$$

*

10

$L_{x12} = \frac{l_x}{\eta_{e1}\eta_{e2}}$, $C_{x12} = \eta_{e1}\eta_{e2}c_x$, $R_{x12} = \frac{r_x}{\eta_{e1}\eta_{e2}}$

Separate freq. response & magnitude:

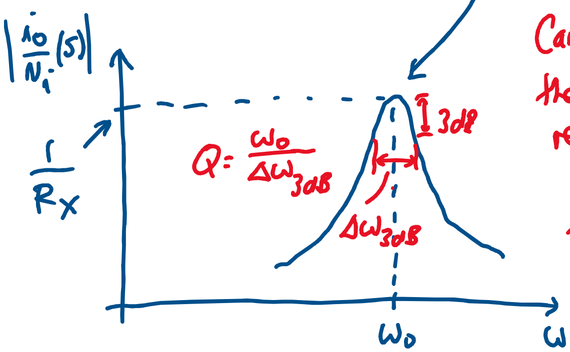
$$\frac{i_o}{v_i}(s) = \frac{1}{sL_x + \frac{1}{sC_x} + R_x} = \frac{s\left(\frac{1}{L_x}\right)}{s^2 + \frac{1}{L_xC_x} + s\left(\frac{R_x}{L_x}\right)}$$

$s = j\omega$

$$\left[\frac{1}{L_xC_x} = \omega_o^2,\quad Q = \frac{\omega_o L_x}{R_x} \rightarrow \frac{R_x}{L_x} = \frac{\omega_o}{Q}\right]$$

$$\frac{i_o}{v_i}(s) = \underbrace{\frac{1}{R_x}}_{\text{Gain Term}} \underbrace{\frac{s\left(\frac{\omega_o}{Q}\right)}{s^2 + s\left(\frac{\omega_o}{Q}\right) + \omega_o^2}}_{\text{Freq. Shaping Term}} = \underbrace{\frac{1}{R_x}}_{\text{resonance magnitude}} H(s)$$

Freq. Shaping Term → Bandpass Biquad



Can just solve the ckt. @ resonance
↓
then multiply the answer by $H(s)$!
↓
w/ Q modification

11

