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EE C247B – ME C218
Introduction to MEMS Design
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Prof. Clark T.-C. Nguyen

Dept. of Electrical Engineering & Computer Sciences
University of California at Berkeley
Berkeley, CA 94720

Lecture Module 9: Energy Methods

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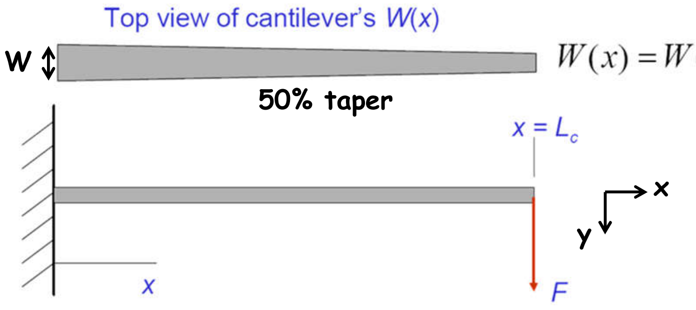
Lecture Outline

- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
    - Virtual Work
    - Energy Formulations
    - Tapered Beam Example

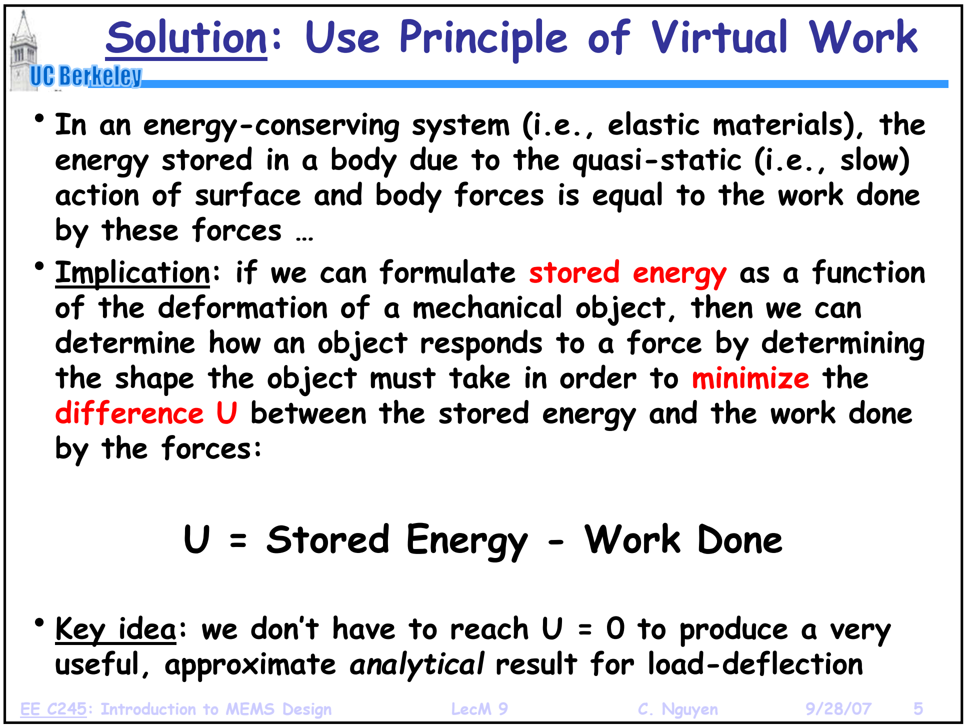
Energy Methods

More General Geometries

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam
- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



$$W(x) = W(1 - \frac{x}{2L_c})$$

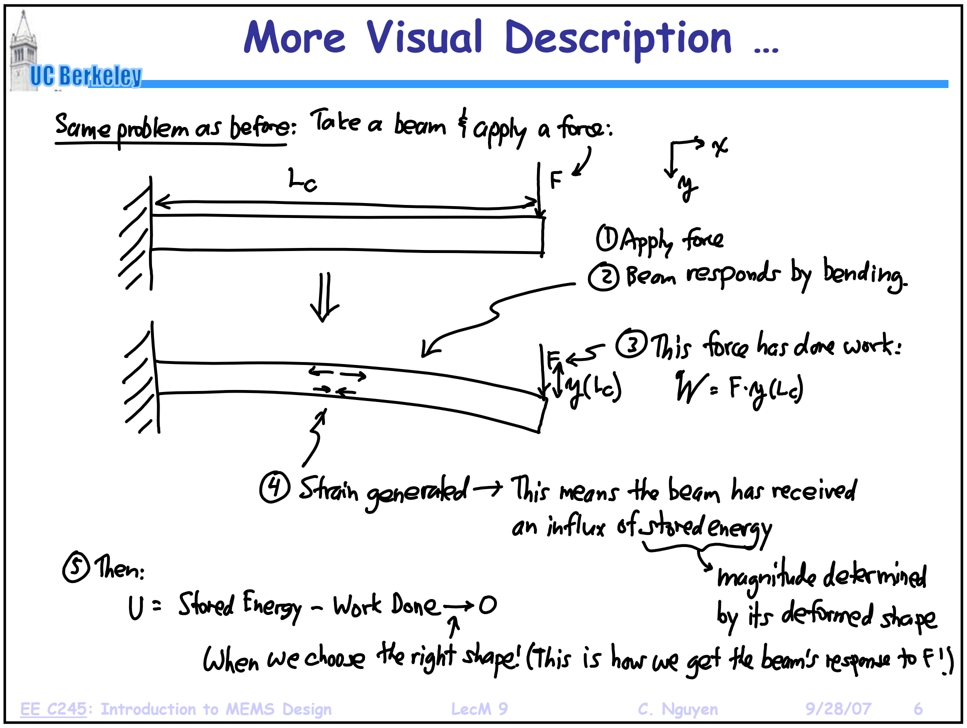


Solution: Use Principle of Virtual Work

- In an energy-conserving system (i.e., elastic materials), the energy stored in a body due to the quasi-static (i.e., slow) action of surface and body forces is equal to the work done by these forces …
- **Implication**: if we can formulate stored energy as a function of the deformation of a mechanical object, then we can determine how an object responds to a force by determining the shape the object must take in order to minimize the difference U between the stored energy and the work done by the forces:

$$U = \text{Stored Energy} - \text{Work Done}$$

- **Key idea**: we don't have to reach U = 0 to produce a very useful, approximate *analytical* result for load-deflection



More Visual Description …

Same problem as before: Take a beam & apply a force:

① Apply force

② Beam responds by bending

③ This force has done work: $W = F \cdot y(L_c)$

④ Strain generated → This means the beam has received an influx of $\underbrace{\text{stored energy}}_{\text{magnitude determined by its deformed shape}}$

⑤ Then:

$$U = \text{Stored Energy} - \text{Work Done} \rightarrow 0$$

When we choose the right shape! (This is how we get the beam's response to F!)

Fundamentals: Energy Density

- Strain energy density: [J/m³]
  - To find work done in straining material

$W(Q) = \int_0^Q \frac{Q}{C} dQ \rightarrow$ charging a capacitor from $0 \rightarrow Q$ takes this much work → stored energy on a capacitor

This is a definition, so really can just say it's a definition

$$w = \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x$$ x-axis normal stress term

← value of strain @ position (x,y,z)

$\sigma_x(\epsilon_x) \rightarrow$ relates stress to strain @ position (x,y,z)

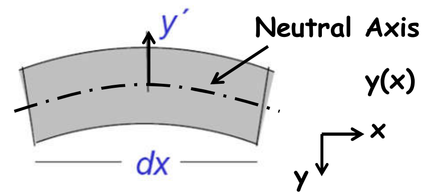
$$\left[\sigma_x = E\epsilon_x\right] \Rightarrow w = \int_0^{\varepsilon_x} E\varepsilon_x d\varepsilon_x = \frac{1}{2}E\varepsilon_x^{\ 2}$$

$W(q_1) = \int_0^{q_1} e(q)dq$ q = displacement, e = effort } Generic Definition of Work

- Total strain energy [J]:
  - Integrate over all strains (normal and shear)

$$\mathcal{W} = \iiint \left(\frac{1}{2}E\left(\varepsilon_x^{\ 2} + \varepsilon_y^{\ 2} + \varepsilon_z^{\ 2}\right) + \frac{1}{2}G\left(\gamma_{xy}^{\ 2} + \gamma_{xz}^{\ 2} + \gamma_{yz}^{\ 2}\right)\right) dV$$

Bending Energy Density



y(x) = transverse displacement of neutral axis

- First, find the bending energy dW_{bend} in an infinitesimal length dx:

W = width

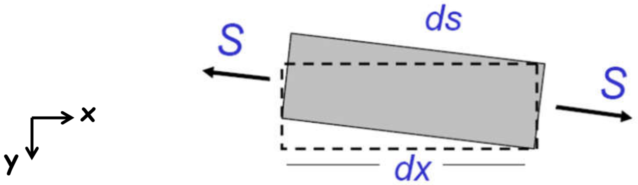
$$dW_{bend} = W dx \int_{-h/2}^{h/2} \frac{1}{2} E \epsilon_x^2(y') dy'$$

$$\left[\frac{1}{R} = \frac{d^2y}{dx^2}, \epsilon_x = \frac{y'}{R}\right] \Rightarrow \epsilon_x(y') = y' \frac{d^2y}{dx^2}$$

$$dW_{bend} = W dx \int_{-h/2}^{h/2} \frac{1}{2} E \left[y' \frac{d^2y}{dx^2}\right]^2 dy' = \frac{1}{2} E \overbrace{\left(\frac{Wh^3}{12}\right)}^{I_z} \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{W_{bend} = \frac{1}{2} E I_z \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}$$

Energy Due to Axial Load



- Strain due to axial load S contributes an energy $d\mathcal{W}_{stretch}$ in length dx, since lengthening of the different element dx (to ds) results in a strain ε_x

Binomial Theorem

$$ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} \approx dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore \epsilon_x = \frac{ds \cdot dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

Axial Strain Energy

$$\left[d\mathcal{W}_{axial} = S\epsilon_x dx = \frac{1}{2}S\left(\frac{dy}{dx}\right)^2 dx\right] \Rightarrow \boxed{\mathcal{W}_{axial} = \frac{1}{2}S\int_0^L \left(\frac{dy}{dx}\right)^2 dx}$$

Shear Strain Energy

$$\boxed{\mathcal{W}_{shear} = \frac{3(EI_z)^2}{4GWh}\int_0^L \left(\frac{d^3y}{dx^3}\right)^2 dx}$$

Shear Modulus

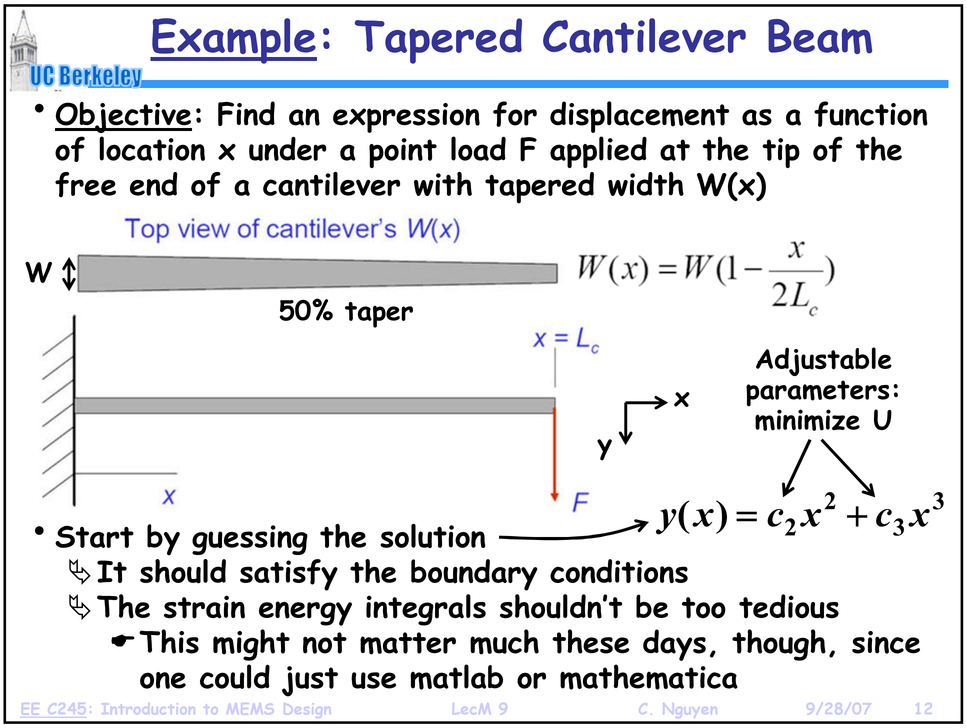
- See W.C. Albert, "Vibrating Quartz Crystal Beam Accelerometer," Proc. ISA Int. Instrumentation Symp., May 1982, pp. 33-44

Applying the Principle of Virtual Work

- **Basic Procedure**:
  - Guess the form of the beam deflection under the applied loads
  - Vary the parameters in the beam deflection function in order to minimize:

$$U = \overbrace{\sum_j W_j}^{\text{Sum strain energies}} - \sum_i \overset{\text{Assumes point load}}{F_i}\,\underset{\text{Displacement at point load}}{u_i}$$

  - Find minima by simply setting derivatives to zero
- See Senturia, pg. 244, for a general expression with distrubuted surface loads and body forces

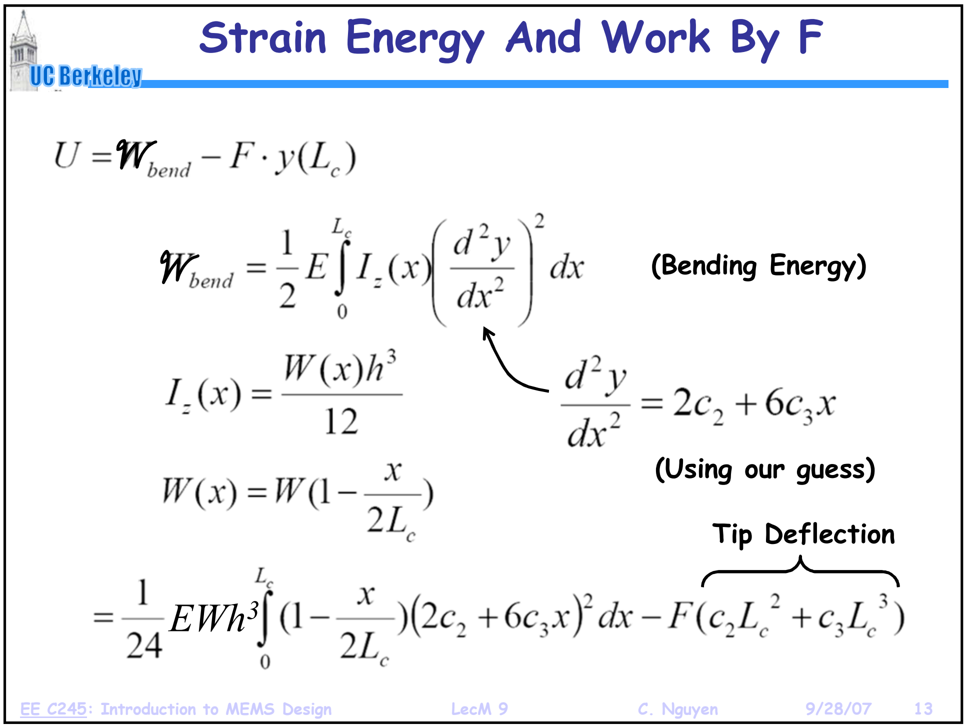


Example: Tapered Cantilever Beam

- **Objective**: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)

$$W(x) = W(1 - \frac{x}{2L_c})$$

- Start by guessing the solution $y(x) = c_2x^2 + c_3x^3$
  - It should satisfy the boundary conditions
  - The strain energy integrals shouldn't be too tedious
    - This might not matter much these days, though, since one could just use matlab or mathematica



Strain Energy And Work By F

$$U = W_{bend} - F \cdot y(L_c)$$

$$W_{bend} = \frac{1}{2} E \int_0^{L_c} I_z(x) \left(\frac{d^2 y}{dx^2} \right)^2 dx$$ (Bending Energy)

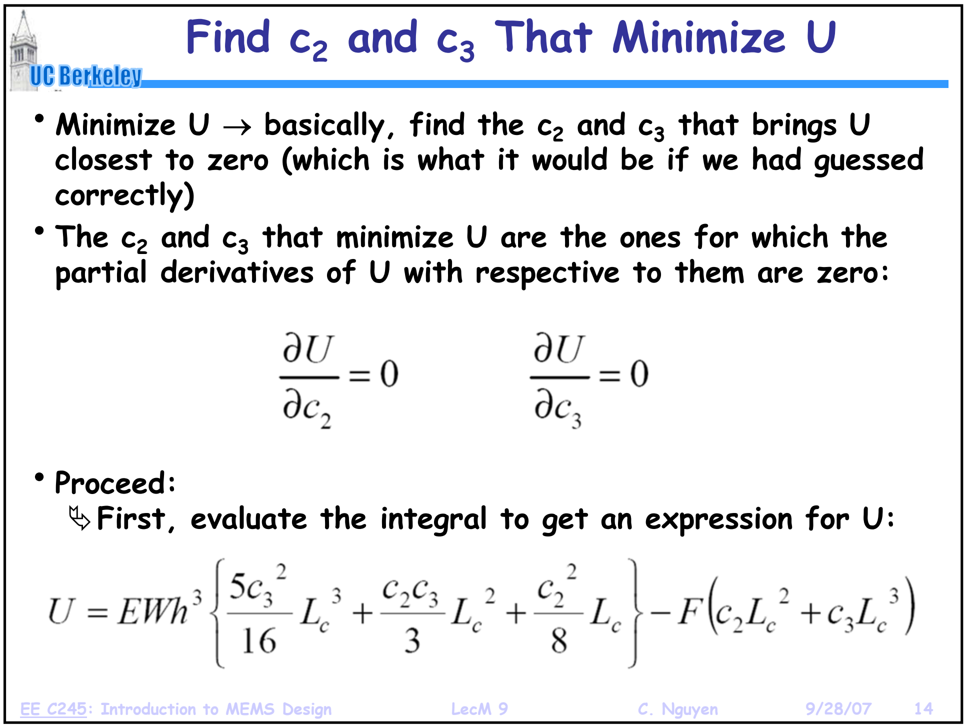
$$I_z(x) = \frac{W(x) h^3}{12}$$

$$\frac{d^2 y}{dx^2} = 2c_2 + 6c_3 x$$

(Using our guess)

$$W(x) = W(1 - \frac{x}{2L_c})$$

$$= \frac{1}{24} E W h^3 \int_0^{L_c} (1 - \frac{x}{2L_c})(2c_2 + 6c_3 x)^2 dx - F\overbrace{(c_2 L_c^{\ 2} + c_3 L_c^{\ 3})}^{\text{Tip Deflection}}$$



Find c_2 and c_3 That Minimize U

- Minimize U → basically, find the c_2 and c_3 that brings U closest to zero (which is what it would be if we had guessed correctly)
- The c_2 and c_3 that minimize U are the ones for which the partial derivatives of U with respective to them are zero:

$$\frac{\partial U}{\partial c_2} = 0 \qquad \frac{\partial U}{\partial c_3} = 0$$

- Proceed:
  - ➥ First, evaluate the integral to get an expression for U:

$$U = EWh^3 \left\{ \frac{5c_3^{\ 2}}{16} L_c^{\ 3} + \frac{c_2 c_3}{3} L_c^{\ 2} + \frac{c_2^{\ 2}}{8} L_c \right\} - F\left(c_2 L_c^{\ 2} + c_3 L_c^{\ 3} \right)$$

Minimize U (cont)

- Evaluate the derivatives and set to zero:

$$\frac{\partial U}{\partial c_2} = 0 = \left(\frac{EWh^3}{3} c_3 - F \right) L_c^{\,2} + \left(\frac{EWh^3}{4} c_2 \right) L_c$$

$$\frac{\partial U}{\partial c_3} = 0 = \left(\frac{5}{8} EWh^3 c_3 - F \right) L_c^{\,3} + \left(\frac{EWh^3}{3} c_2 \right) L_c^{\,2}$$

- Solve the simultaneous equations to get c_2 and c_3:

$$\boxed{c_2 = \left(\frac{84}{13} \right) \frac{FL_c}{EWh^3} \qquad c_3 = -\left(\frac{24}{13} \right) \frac{F}{EWh^3}}$$

The Virtual Work-Derived Solution

- And the solution:

$$y(x) = \left(\frac{24F}{13EWh^3} \right) \left(\left(\frac{7}{2} \right) L_c - x \right) x^2$$

- Solve for tip deflection and obtain the spring constant:

$$y(L_c) = \left(\frac{24F}{13EWh^3} \right) \left(\frac{5}{2} \right) L_c^{\,3} \qquad k_c = F / y(L_c) = \left(\frac{13EWh^3}{60L_c^{\,3}} \right)$$

- Compare with previous solution for constant-width cantilever beam (using Euler theory):

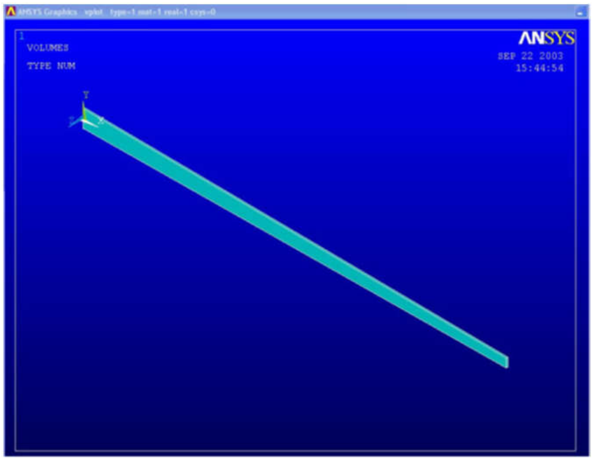
$$y(L_c) = \left(\frac{4F}{EWh^3} \right) L_c^{\,3} \longrightarrow$$

13% smaller than tapered-width case

Comparison With Finite Element Simulation

- Below: ANSYS finite element model with

L = 500 μm W_{base} = 20 μm E = 170 GPa
h = 2 μm W_{tip} = 10 μm



- Result: (from static analysis)
  - ↳ k = 0.471 μN/m
- This matches the result from energy minimization to 3 significant figures

Need a Better Approximation?

- Add more terms to the polynomial
- Add other strain energy terms:
  - ↳ Shear: more significant as the beam gets shorter
  - ↳ Axial: more significant as deflections become larger
- Both of the above remedies make the math more complex, so encourage the use of math software, such as Mathematica, Matlab, or Maple
- Finite element analysis is really just energy minimization
- If this is the case, then why ever use energy minimization analytically (i.e., by hand)?
  - ↳ Analytical expressions, even approximate ones, give insight into parameter dependencies that FEA cannot
  - ↳ Can compare the importance of different terms
  - ↳ Should use in tandem with FEA for design