EE290F Fourier Optics Fall 2001
Final Exam Solutions

1. a) For a periodic grating with period $L$, the transmission can be represented by the Fourier series:

$$t_A(\xi) = \sum_{\substack{k=-\infty \\ k \text{ integer}}}^{\infty} c_k e^{j 2\pi k \xi / L}$$

where $$c_k = \frac{1}{L} \int_{-L/2}^{L/2} t_A(\xi) e^{-j 2\pi k \xi / L} d\xi$$

$c_k$ is the amplitude of the $k$th diffracted order. $|c_k|^2$ is the diffraction efficiency (see Problem set 3, #2)

If we translate the grating by a distance $\xi_s$, then

$$c_k' = \frac{1}{L} \int_{-L/2}^{L/2} t_A(\xi - \xi_s) e^{-j 2\pi k \xi / L} d\xi$$

let $x = \xi - \xi_s$

$$= \frac{1}{L} \int_{-(L+\xi_s)/2}^{(L+\xi_s)/2} t_A(x) e^{-j 2\pi k (x + \xi_s)/L} dx$$

$$= e^{-j 2\pi k \xi_s / L} \frac{1}{L} \int_{-(L+\xi_s)/2}^{(L+\xi_s)/2} t_A(x) e^{-j 2\pi k x / L} dx$$

$$= e^{-j 2\pi k \xi_s / L} c_k$$

Each order is phase shifted by $e^{-j 2\pi k \xi_s / L}$. If $\xi_s = L/2$, the shift is $2\pi k / 2 = \pi k$. So the relative phase shift between adjacent orders is $\pi$.

b) Write the amplitude transmittance function in the region $|\xi| \le L/2$ as

$$t_A(\xi) = \left(\tfrac{1}{2} - t_m\right) \mathrm{rect}\left(\xi/L\right) + 2t_m\, \mathrm{rect}\left(\frac{\xi}{L/2}\right)$$

Then $$C_k = \frac{1}{L} \mathcal{F}\left\{\left(\tfrac{1}{2} - t_m\right) \mathrm{rect}\left(\xi/L\right) + 2t_m\, \mathrm{rect}\left(\frac{\xi}{L/2}\right)\right\}_{f_X = k/L}$$

$$= \left(\tfrac{1}{2} - t_m\right) \mathrm{sinc}(k) + t_m\, \mathrm{sinc}\left(\frac{k}{2}\right)$$

The first order efficiency is

$$|C_1|^2 = \left|\left(\tfrac{1}{2} - t_m\right) \underbrace{\mathrm{sinc}(1)}_{0} + t_m \underbrace{\mathrm{sinc}\left(\tfrac{1}{2}\right)}_{2/\pi}\right|^2$$

$$= \left(\frac{2t_m}{\pi}\right)^2 = \boxed{\frac{4t_m^2}{\pi^2}}$$

The maximum physically allowed value for $t_m$ is $1/2$

$$\boxed{t_m = \tfrac{1}{2}}$$

c) We can write

$$t_A(x) = 1 - \left[\left(1 - e^{j\phi}\right) \times (\text{square wave})\right]$$

$$= 1 - \left[\left(1 - e^{j\phi}\right) \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n x/L}\right]$$

with $$c_n = \frac{1}{L} \mathcal{F}\left\{\mathrm{rect}\, \frac{x}{L/2}\right\}_{f_X = n/L} = \frac{1}{2} \mathrm{sinc}\left(\frac{n}{2}\right)$$

Then,

$$\mathcal{F}[t_A(x)] = \delta(f_x) - (1-e^{j\phi}) \sum_{n=-\infty}^{\infty} \frac{1}{2} \operatorname{sinc}\left(\frac{n}{2}\right) \mathcal{F}\{e^{j2\pi nx/L}\}$$

$$= \delta(f_x) - (1-e^{j\phi}) \sum_{n=-\infty}^{\infty} \frac{1}{2} \operatorname{sinc}\left(\frac{n}{2}\right) \delta\left(f_x - \frac{n}{2}\right)$$

The first order term is $(1-e^{j\phi})\frac{1}{2}\operatorname{sinc}\frac{1}{2}$, and the efficiency is

$$\eta_1 = |1-e^{j\phi}|^2 \left[\frac{1}{2}\operatorname{sinc}\frac{1}{2}\right]^2$$

$$= \frac{1}{4}\left(\frac{2}{\pi}\right)^2 (2 - 2\cos\phi) = \frac{2}{\pi^2}(1-\cos\phi)$$

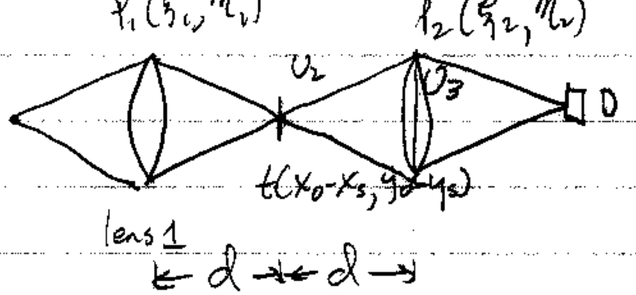
To maximize, we want $\cos\phi = -1$, so $\boxed{\phi = \pi}$

The third order term is

$$\eta_3 = |1-e^{j\phi}|^2 \left[\frac{1}{2}\operatorname{sinc}\left(\frac{3}{2}\right)\right]^2.$$

This maximizes for the same value, $\boxed{\phi = \pi}$

2.




let $h_1(x_0, y_0)$ be the PSF from lens 1

the field just after the object is

$$U_2(x_0, y_0; x_s, y_s) = h_1(x_0, y_0)\, t(x_0 - x_s, y_0 - y_s)$$

where $x_s, y_s$ is the scan position of the object.

The field at the $P_2$ plane is (neglecting the phase factor)

$$U_3(\xi_2, \eta_2; x_s, y_s) = \iint h_1(x_0, y_0)\, t(x_0 - x_s, y_0 - y_s) \exp \frac{jk}{d}(\xi_2 x_0 + \eta_2 y_0)\, dx_0 dy_0$$

Define $P_2(\xi_2, \eta_2)$ as the pupil fn for lens 2,
Then the detected signal is proportional to

$$I(x_s, y_s) = \iint |U_3(\xi_2, \eta_2; x_s, y_s) P_2(\xi_2, \eta_2)|^2 d\xi_2 d\eta_2$$

Then

$$I(x_s, y_s) = \iiiint h_1(x_o, y_o) h_1^*(x_o', y_o') t(x_o - x_s, y_o - y_s)$$
$$t_o^*(x_o' - x_s, y_o' - y_s) \times$$
$$\iint P_2(\xi_2, \eta_2) P_2^*(\xi_2, \eta_2) \exp \frac{jk}{d}(\xi_2 x_o + \eta_2 y_o)$$
$$\exp -\frac{jk}{d}(\xi_2 x_o' + \eta_2 y_o') \, dx_o dy_o dx_o' dy_o' d\xi_2 d\eta_2$$

$$= \iiiint h_1(x_o, y_o) h_1^*(x_o', y_o') t(x_o - x_s, y_o - y_s) t_o^*(x_o' - x_s, y_o' - y_s)$$
$$g_2(x_o - x_o', y_o - y_o') \, dx_o dy_o dx_o' dy_o' \qquad \text{(A)}$$

$$g_2(x, y) = \iint |P_2(\xi_2, \eta_2)|^2 \exp \frac{jk}{d}(x \xi_2 + y \eta_2) \, d\xi_2 d\eta_2 \qquad \text{(B)}$$

which is the spread function for $|P_2|^2$

Limit ① detector is infinitely large with constant sensitivity.

Then $g_2(x_o - x_o', y_o - y_o') = \delta(x_o - x_o', y_o - y_o')$

$$I(x_s, y_o) = \iint |h_1(x_o, y_o)|^2 \cdot |t(x_o - x_s, y_o - y_s)|^2 dx_o dy_o$$

$$= |h_1|^2 \circledast |t|^2$$

limit ② detector is extremely small

$g_2 \Rightarrow$ constant

$$I(x_s, y_s) = \left| \iint h_1(x_o, y_o) t(x_o - x_s, y_o - y_s) dx_o dy_o \right|^2$$

$$= |h_1 \circledast t|^2$$

Field after object is

$$U_2(x_0, y_0; x_s, y_s) = h_1(x_0, y_0)\, t(x_0 - x_s, y_0 - y_s)$$

Image at detector plane is convolution of $U_2$ with $h_1$

$$U_D(x_2, y_2; x_s, y_s) = \iint h_1(x_0, y_0)\, t(x_0 - x_s, y_0 - y_s)\, h_1\!\left(\frac{x_2}{M} - x_0, \frac{y_2}{M} - y_0\right) dx_0\, dy_0$$

Select at $x_2 = y_2 = 0$. Note $M = -1$. So if $h_1$ is even: $h_1(x,y) = h_1(-x,-y)$ then:

$$I_D(x_s, y_s) = \left| \iint h_1^2(x_0, y_0)\, t(x_0 - x_s, y_0 - y_s)\, dx_0\, dy_0 \right|^2$$

$$I_D(x_s, y_s) = \left| h_1^2 \otimes t \right|^2$$

3. Let $t_A(x_0, y_0)$ be the object transmittance

The field doe to the object at the detector plane is

$$U_o(x,y) = \frac{1}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \mathcal{F}[t_A]_{\substack{f_x = x/\lambda z \\ f_y = y/\lambda z}}$$

Let the reference field point be at $(0, y_r)$ so the reference field at the detector is

$$U_r(x,y) = \frac{A}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} e^{-j\frac{k}{z} y_r y}$$

The intensity is then (let $u = \frac{x}{\lambda z}$, $v = \frac{y}{\lambda z}$)

$$I(x,y) = \frac{1}{\lambda^2 z^2} \left| A e^{-j2\pi v y_r} + \mathcal{F}[t_A] \right|^2$$

$$= \frac{1}{\lambda^2 z^2} \left[ |A|^2 + |\mathcal{F}[t_A]|^2 + A e^{-j2\pi v y_r} \mathcal{F}^*[t_A] + A^* e^{j2\pi v y_r} \mathcal{F}[t_A] \right]$$

If we take the Fourier Transform of $I(x,y)$, we get:

$$\mathcal{F}[I(x,y)] = \frac{1}{\lambda^2 z^2} \left\{ \mathcal{F}[|A|^2] + \mathcal{F}\left[ |\mathcal{F}[t_A]|^2 \right] + \mathcal{F}\left[ A e^{-j2\pi y_r v} \mathcal{F}^*[t_A] \right] + \mathcal{F}\left[ A^* e^{j2\pi y_r v} \mathcal{F}[t_A] \right] \right\}$$

$$= \frac{1}{\lambda^2 z^2} \left[ |A|^2 \delta(x,y) + \mathcal{F}\mathcal{F}[t_A] \circledast \mathcal{F}\mathcal{F}^*[t_A] + A \mathcal{F}\mathcal{F}^*[t_A] \circledast \delta(x, y+y_r) + A^* \mathcal{F}\mathcal{F}[t_A] \circledast \delta(x, y-y_r) \right.$$

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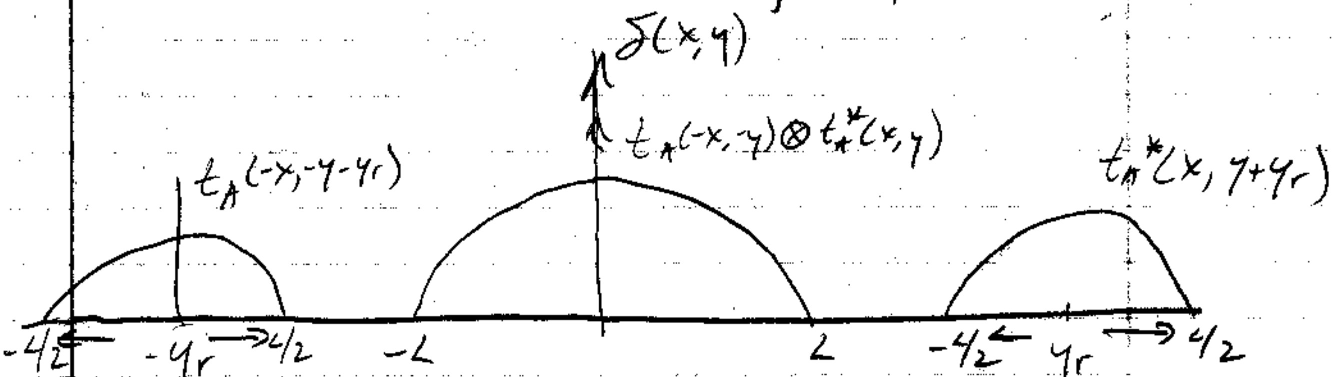
Continuing

$$\mathcal{F}[I(x,y)] = \frac{1}{(\lambda z)^2}\Big[|A|^2\delta(x,y) + t_A(-x,y)\otimes t_A^*(x,y)$$
$$+ A t_A^*(x,y)\otimes\delta(x,y+y_r)$$
$$+ A^* t_A(-x,-y)\otimes\delta(x,y-y_r)\Big]$$
$$= \frac{1}{\lambda^2 z^2}\Big[|A|^2\delta(x,y) + t_A(-x,-y)\otimes t_A^*(x,y)$$
$$+ A\, t_A^*(x,y+y_r) + A^* t_A(-x,-y-y_r)\Big]$$

So we can recover an inverted, shifted version of $t_A$ from the 4th term, if it can be separated from the other terms.

b) The FT of the intensity appears as follows




The minimum value for $y_r$ is $3L/2$

c) The highest spatial frequency in the object that is captured by the detector is $f_{max} = X/2\lambda z$ where $X$ is the size of the detector. The spatial resolution is $1/2f_{max} = \lambda z/X$