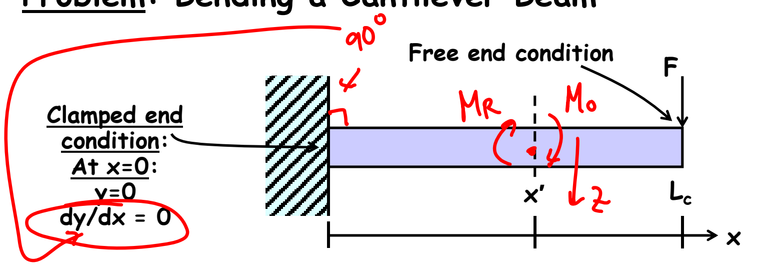
Lecture 15: Stress Gradients

- Announcements:
- Reminder: HW#4 due on Tuesday, Oct. 19
- Midterm will be Tuesday, Nov. 2
- ------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- ----------------------------------
- Last Time:

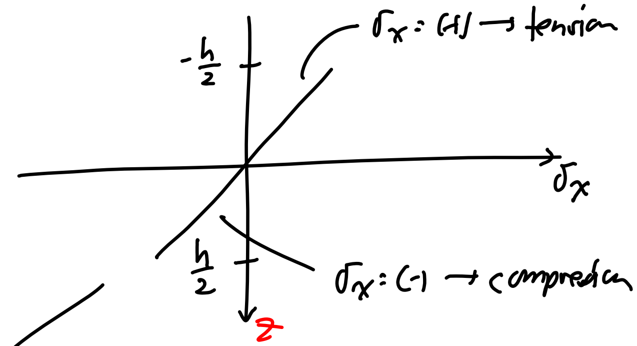
Problem: Bending a Cantilever Beam



- Objective: Find relation between tip deflection $y(x=L_c)$ and applied load F
- Assumptions:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible

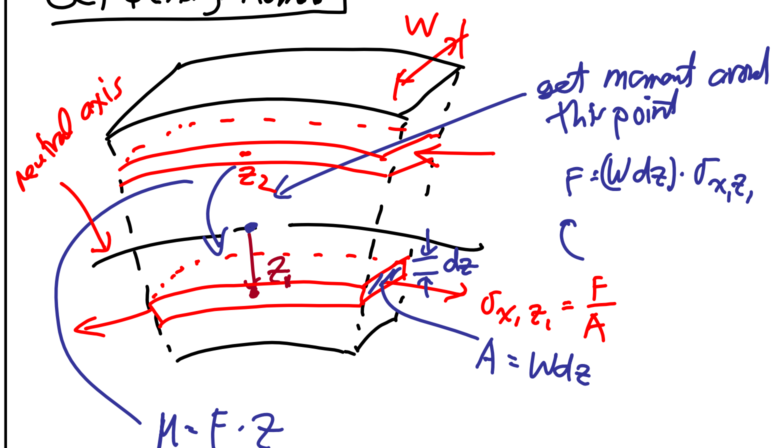
Of course, there is a corresponding axial stress:

$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$



This gradient in stress → generates a bending moment! in response to the original applied moment!

Get Bending Moment



⇒ integrate stress over the thickness of the beam:

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} \Big[\underbrace{(W dz)\,\sigma_x}_{\text{force}}\Big] \cdot z$$

$$= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{EWz^2}{R}\,dz \rightarrow \boxed{M = -\left(\frac{1}{12}Wh^3\right)\frac{E}{R}}$$

$$\left[\sigma_x = -\frac{zE}{R}\right]$$

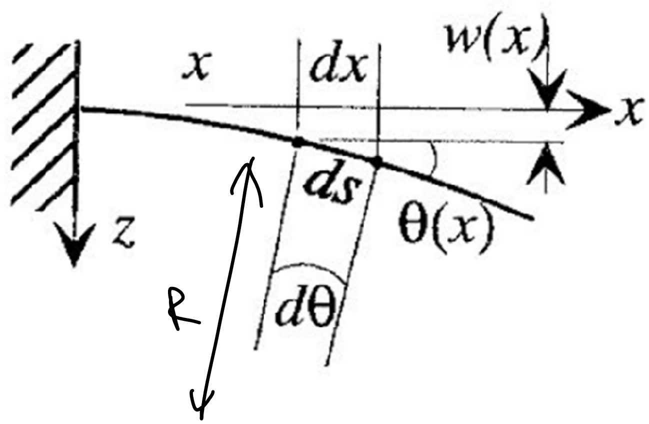
$\frac{1}{12}Wh^3 = I \triangleq$ Moment of Inertia

Internal Moment ($= M_R$)

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

radius of curvature

Differential Beam Bending Equation



Write out some geometric relationships:

⇒ use small angle approx.

$$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \rightarrow ds \approx dx$$

$\tan\theta = \frac{dw}{dx}$: slope of the beam @ this pt. $\rightarrow \theta \approx \frac{dw}{dx}$ (1)

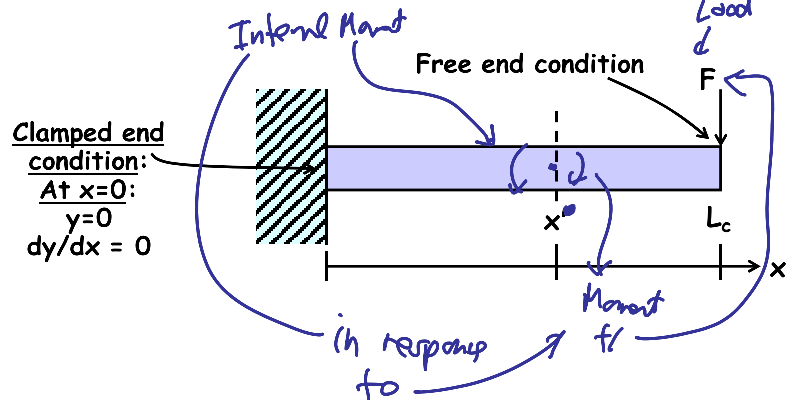
$$ds = R\,d\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \rightarrow \frac{1}{R} = \frac{d\theta}{dx} \quad (2)$$

Inserting (1) into (2):

$$\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$$

Differential Eq. for Small Angle Beam Bending

Cantilever Beam w/ Concentrated Load



Internal Moment @ position x: $M = -F(L-x)$

Thus:

$$\frac{d^2 w}{dx^2} = \frac{F}{EI}(L-x)$$

w/ Clamped End B.C.: $w(x=0)=0$, $\frac{dw}{dx}(x=0)=0$

Free End B.C.: none

Solve to get expression for w

→ use Laplace; or use a trial solution:

$$w = A + Bx + Cx^2 + Dx^3,$$

then apply B.C.'s

$$\boxed{w = \frac{FL}{2EI}x^2\left(1-\frac{x}{3L}\right)}$$

[Deflection @ x due to a point load F applied @ $x = L$]

Maximum deflection @ $x = L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \left(\frac{3EI}{L^3}\right)w(x=L)$$

$$= k_c\, w(x=L)$$

"cantilever"

$$\boxed{k_c = \frac{3EI}{L^3}} \triangleq \text{stiffness @ location } \underline{x = L}$$

in general, stiffness is a function of location

$$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \boxed{k_c = \frac{1}{4}EW\frac{h^3}{L^3}}$$

Ex: $L = 100\mu m$, $W = 2\mu m$, $h = 2\mu m$

polysilicon → $E = 150\,GPa$

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = \underline{0.6\ N/m}$$

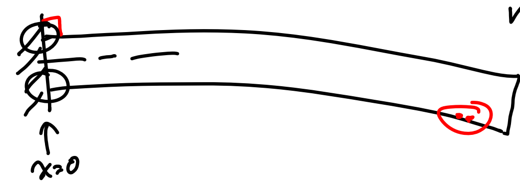
Maximum Stress in a Bent Cantilever

From before, the radius of curvature is given by

$$\frac{1}{R} = \frac{d^2 w}{dx^2} = \frac{F}{EI}(L-x)$$

→ where is $\frac{1}{R}$ maximized? → where R is minimized...

@ $x=0$



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$$[x=0] \Rightarrow \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$$

Strain is maximized: $\epsilon_{max} = \frac{h}{2}\frac{FL}{EI}$

① At top surface → tensile

② At the bottom surface → compressive

$$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \epsilon_{max} = \frac{\cancel{h}}{2}\frac{FL}{E}\left(\frac{12}{Wh^{\cancel{3}\,2}}\right) = \frac{6L}{EWh^2}F$$

$$\therefore \boxed{\sigma_{max} = \epsilon_{max}E = \frac{6L}{Wh^2}F}$$

(maximum strain in a bent cantilever)

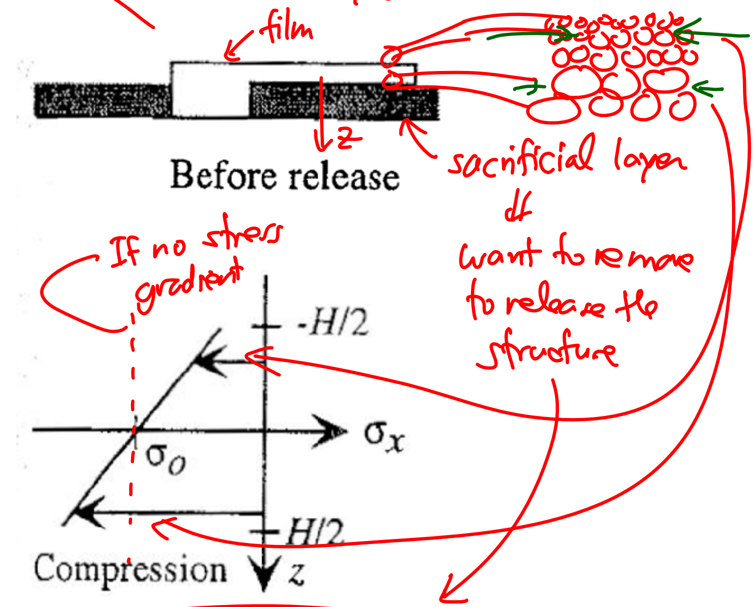
Stress Gradients in Cantilevers

Achieved by:

① Deposit film @ high T

② Cool it down

③ Pattern & etch into cantilever



Stress before release

→ want to remove to release the structure

release the structure ④

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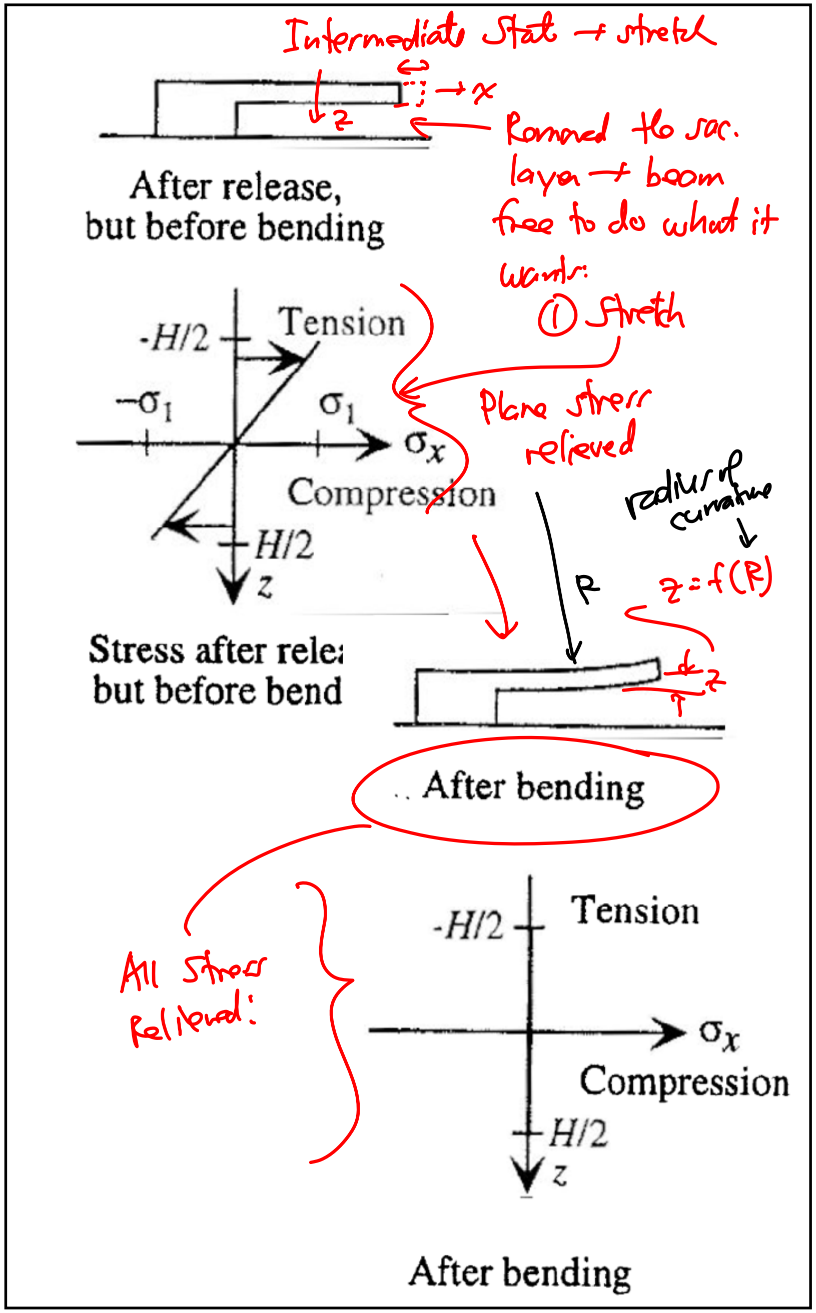

Stress Gradients in Cantilevers → Bending

Find the radius of curvature.

Prior to release, axial stress: $\sigma = \sigma_0 - \frac{\sigma_1}{(H/2)} z$

The internal moment:

$$M_x = \int_{-\frac{H}{2}}^{\frac{H}{2}} [(W dz)\cdot\sigma]\cdot z = W\int_{-\frac{H}{2}}^{\frac{H}{2}}\left(z\sigma_0 - \frac{\sigma_1 z^2}{(H/2)}\right)dz$$

$$= W\left(\frac{1}{2}\sigma_0 z^2 - \frac{2\sigma_1 z^3}{3H}\right)\Bigg|_{-H/2}^{H/2}$$

$$= W\left(\cancel{\frac{1}{2}\sigma_0\frac{H^2}{4}} - \frac{2}{3}\sigma_1\frac{H^2}{8} - \cancel{\frac{1}{2}\sigma_0\frac{H^2}{4}} - \frac{2\sigma_1 H^2}{3(8)}\right)$$

Average stress nulls out.

$$\boxed{M_x = -\frac{1}{6}\sigma_1 W H^2}$$

Thus, the radius of curvature:

$$\frac{1}{R} = -\frac{M_x}{E'I} \rightarrow R = \frac{E'I}{M_x} = \frac{1}{2}\frac{E'H}{\sigma_1}$$

Biaxial Modulus

$[I = \frac{1}{12}Wh^3]$

$$\boxed{R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1}}$$

Radius of Curvature for a Cantilever w/ a Stress Gradient

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