Lecture 17: Energy Methods

- Announcements:
- Midterm will be Tuesday, Nov. 2
- I will not be here on Tuesday, Oct. 26
- We will make up the lecture one day earlier, on Monday, Oct. 25, from 5:30-7 p.m., in 3107 Etcheverry (basically, I will lecture during your discussion section)
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- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
  - Estimating Resonance Frequency
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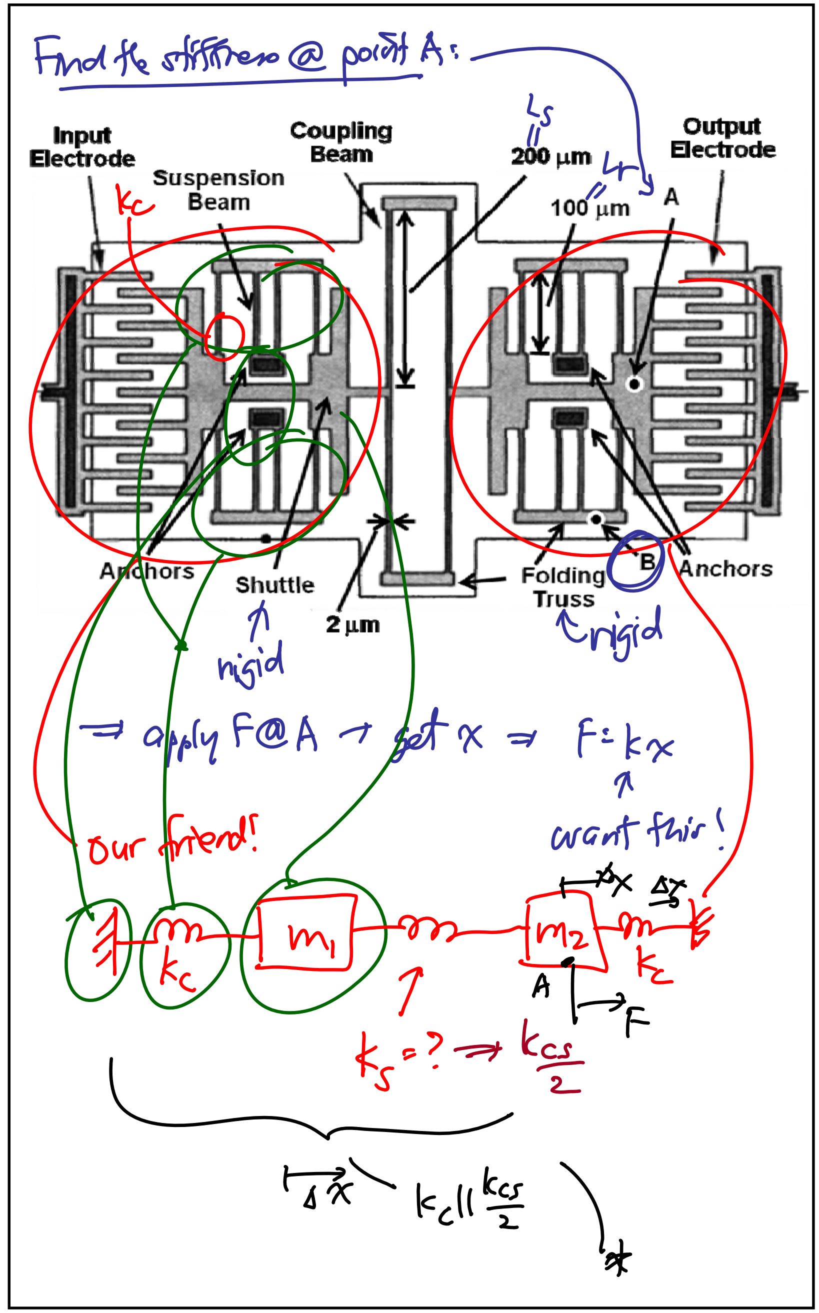
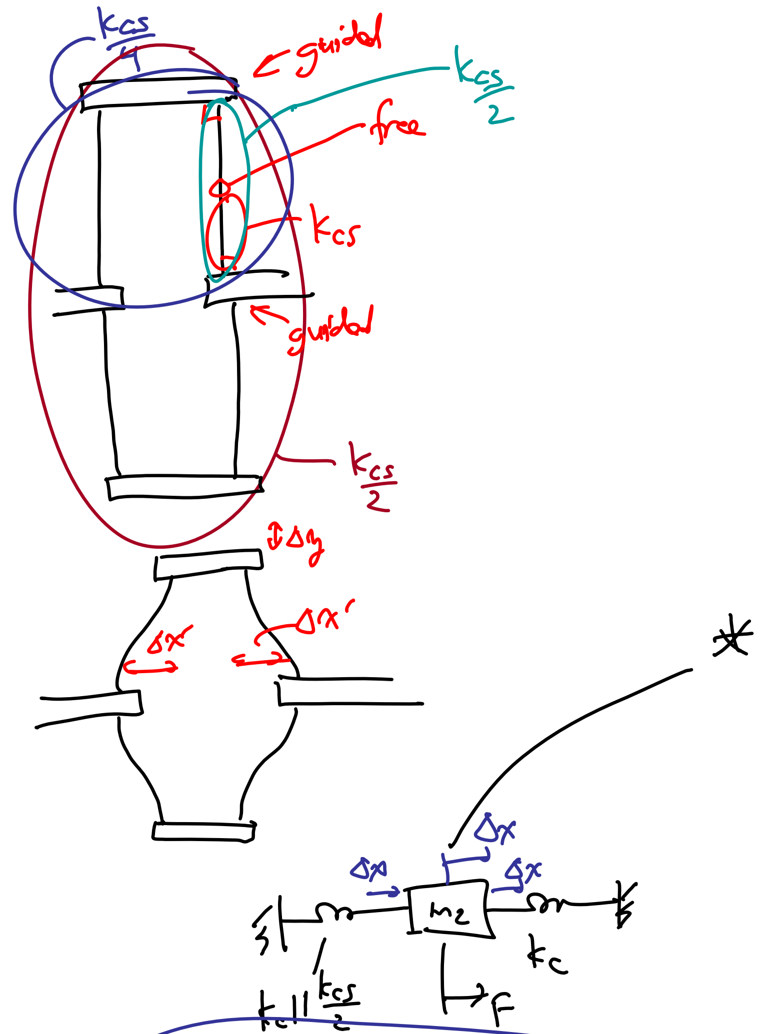
- Last Time:
- Went through beam combos
- Practiced reducing complex mechanical circuits to simpler ones that can be analyzed quickly

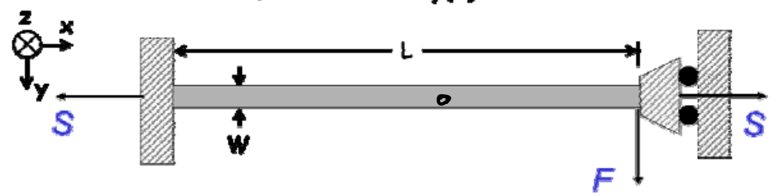




$$k_{xA} = k_c + k_c \| \frac{k_{cs}}{2}$$

where $k_c = \frac{24EI_z}{L_r^3}$ and $k_{cs} = \frac{24EI_z}{L_s^3}$

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$

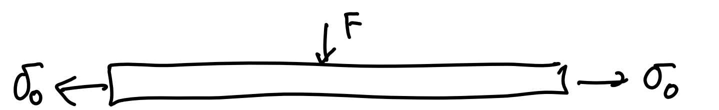


Governing differential equation: (Euler Beam Equation)

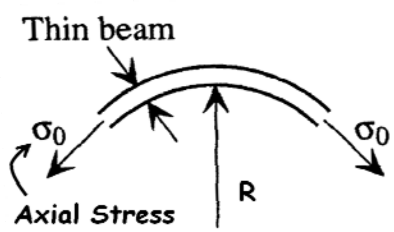
$$EI_z \frac{d^4y}{dx^4} - \underset{\text{Axial Load}}{S} \frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

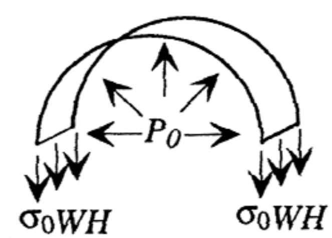
Heuristic Derivation for the Euler Beam Equation

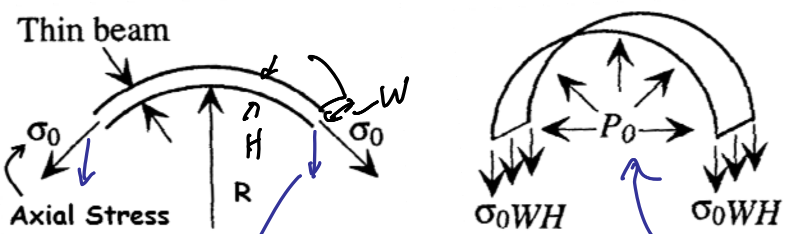
Consider a straight unflexed beam under an axial stress:



@ "0" displacement ⇒ no effect on stiffness $\hat{y}$-directed







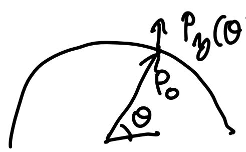
w/ bent beam, σ_0 has a z-directed component → affects stiffness!

Upward pressure P_0 to counteract the downward force from → to keep everything in static equilibrium

For ease of analysis, assume the beam is bent to angle π

Downward Vertical Force $= \underline{2\sigma_0 WH}$

Upward Force due to P_0:



$$F_u = \int_0^{\pi} (P_0 \sin\theta)\, W\, (R\, d\theta)$$

$$= -P_0 W R \cos\theta \Big|_0^{\pi}$$

$$= \underline{2RWP_0}$$

[Equilibrium] $\Rightarrow 2RWP_0 = 2\sigma_0 WH \rightarrow P_0 = \dfrac{\sigma_0 H}{R}$

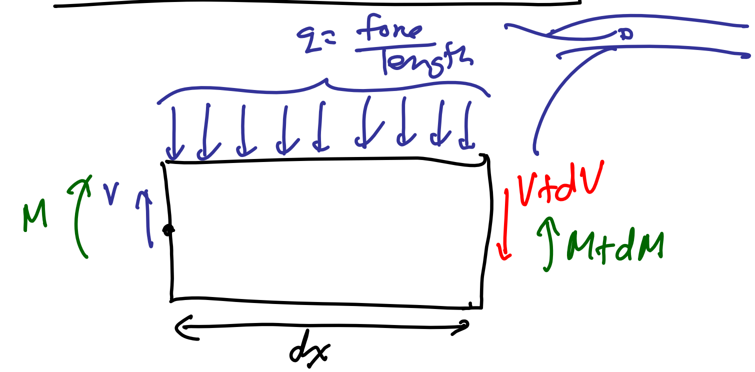
$\left[q_0 = \dfrac{\text{beam load}}{\text{unit length}} = P_0 W \text{ (width)}, \ \dfrac{1}{R} = \dfrac{d^2 w}{dx^2} \right] \Rightarrow$

$q_0 = \sigma_0 WH \dfrac{d^2 w}{dx^2}$ (beam displacement) $\Rightarrow$ generalize to the case of small displacements & small angles

Using the differential beam bending equation:

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \longrightarrow \frac{d^4 w}{dx^4} = -\frac{q}{EI} \quad \left(\frac{\text{load}}{\text{unit length}}\right)$$

Relationships Between Forces on a Fully Loaded Differential Beam Element



[Total static equilibrium] ⇒ total force = 0

F_T = total force = $q\,dx + (\cancel{V} + dV) - \cancel{V} = 0$

$$\therefore \boxed{\frac{dV}{dx} = -q} \qquad (1)$$

⇒ also, total moment w/r to the left hand edge = 0

$$M_T = (\cancel{M} + dM) - \cancel{M} - (V + \cancel{dV})dx - \cancel{\underbrace{\frac{q\,dx}{2}dx}_{\int_0^{dx}(q\,du)u = \frac{1}{2}q\,dx^2}} = 0$$

[Neglect products of differentials] ⇒

$$dM - V\,dx = 0 \rightarrow \boxed{\frac{dM}{dx} = V} \qquad (2)$$

Using (1) & (2):

$$* \left[\frac{d^2M}{dx^2} = \frac{dV}{dx} = -q\right]$$

$$EI\frac{d^4w}{dx^4} = q + q_0$$

q ← external load

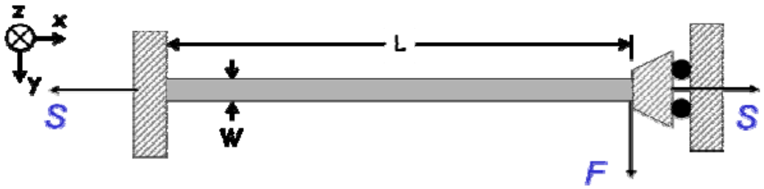
q_0 ← equiv. load accounting for axial stress contributions to the bending stiffness

$$\left[q_0 = \sigma_0 WH\frac{d^2w}{dx^2}\right] \Rightarrow \boxed{EI\frac{d^4w}{dx^4} - \underbrace{(\sigma_0 WH)}_{\text{tension in the beam} = S}\frac{d^2w}{dx^2} = q}$$

S ← a force

Euler Beam Equation

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$



Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - \underset{\text{Axial Load}}{S} \frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko:

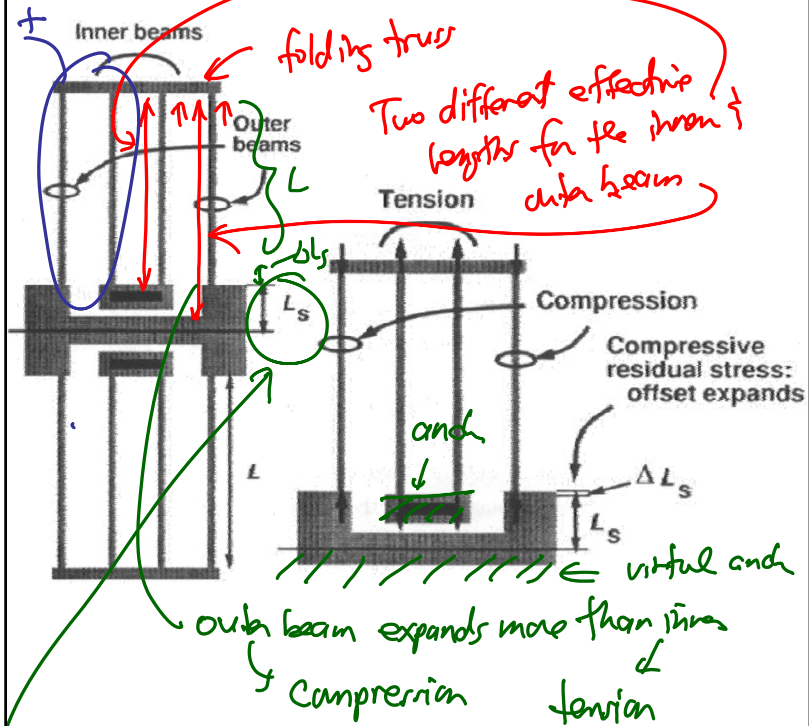
$S > 0$ (tension) $$k^{-1} = \frac{pL - 2\tanh(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

$S < 0$ (compression)

$$k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

where $p = \sqrt{\frac{|S|}{EI_z}}$

Folded-Beams Do Not Completely Relieve Stress



① If the polysilicon strain is ϵ_r, then shuttle expands by $\Delta L_s = \epsilon_r L_s$

② This applies a load to the beams, $\Delta L = \Delta L_s$

③ Beam strain:

$$\epsilon_b = \frac{\Delta L}{L} = \frac{\Delta L_s}{L} = \pm\epsilon_r \frac{L_s}{L}$$

*

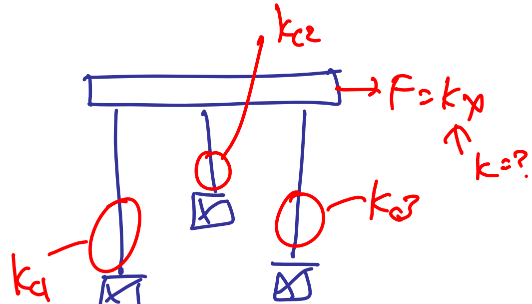
* Shear Force: $S = \pm E\epsilon_L \left(\frac{L_s}{L}\right) wh$ (whole)

Spring Constant for folded Beam Structure:

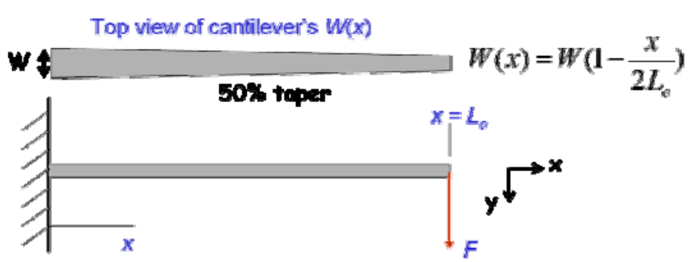
$$k = 4(k_{cant}^{-1} + k_{tens}^{-1})^{-1}$$

$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{p|S|} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$

outer & inner beam in series = $k_{cant} \| k_{ten}$



- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam
- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



$\boxed{\text{Fundamentals: Energy Density}}$

General Definition Work:

$$W(q_1) = \int_0^{q_1} e(q)\,dq \qquad q\text{: displacement} \quad e\text{: effort}$$

for EEs: $W(Q) = \int_0^Q \frac{Q}{C}\,dQ$

Strain Energy Density:

$$W = \int_0^{\epsilon_x} \sigma_x \, d\epsilon_x$$

ϵ_x ← value of strain @ position (x, y, z)

$\sigma_x(\epsilon_x)$ → relates stress to strain @ position (x, y, z)

$[\sigma_x = E\epsilon_x]$ ⇒

$$W = \int_0^{\epsilon_x} E\epsilon_x \, d\epsilon_x = \frac{1}{2}E\epsilon_x^2$$

Total Strain Energy [J]: (for 3D)

$$W = \iiint \left(\frac{1}{2}E(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \frac{1}{2}G(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2)\right) dV$$

G: shear modulus