Lecture 18: Energy Methods

- **Announcements**:
- Note: this is a Monday, Oct. 26, 5:30-7 p.m., lecture in place of Tuesday, Oct. 26, when I will be traveling
- Reminder: Midterm will be Tuesday, Nov. 2
- Midterm Info Sheet passed out
- No office hours on Wednesday
- ----------------------------------
- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
  - Estimating Resonance Frequency
- ----------------------------------
- **Last Time**:

$\boxed{\text{Fundamentals: Energy Density}}$

General Definition Work:

$$W(q_1) = \int_0^{q_1} e(q)\,dq$$

q: displacement
e: effort

for EE: $W(Q) = \int_0^Q \frac{Q}{C}\,dQ$

Strain Energy Density

$$w = \int_0^{\epsilon_x} \sigma_x \, d\epsilon_x$$

ϵ_x ← value of strain @ position (x,y,z)

$\sigma_x(\epsilon_x)$ → relates stress to strain @ position (x,y,z)

$[\sigma_x = E\epsilon_x]$ ⇒

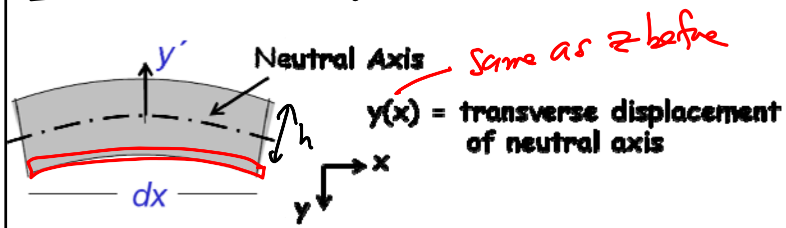
$$w = \int_0^{\epsilon_x} E\epsilon_x \, d\epsilon_x = \frac{1}{2}E\epsilon_x^2$$

Total Strain Energy [J]: (for 3D)

$$W = \iiint \left(\frac{1}{2}E(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \frac{1}{2}G(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2) \right) dV$$

G ↑ shear modulus

Bending Energy Density



First, find the bending energy dW_{bend} in an infinitesimal length dx:

$$dW_{bend} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} E \epsilon_x^2(y') \, dy'$$

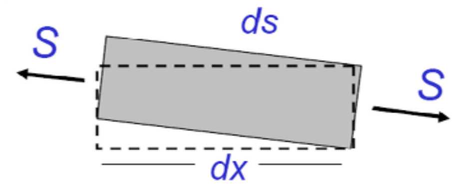
$$\left[\frac{1}{R} = \frac{d^2y}{dx^2}, \; \epsilon_x = \frac{y'}{R} \right] \Rightarrow \epsilon_x(y') = y' \frac{d^2y}{dx^2}$$

$$dW_{bend} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} E \left[y' \frac{d^2y}{dx^2} \right]^2 dy'$$

$$= \frac{1}{2} E \underbrace{\left(\frac{Wh^3}{12} \right)}_{I_z} \left(\frac{d^2y}{dx^2} \right)^2 dx$$

$$\therefore \boxed{W_{bend} = \frac{1}{2} E I_z \int_0^L \left(\frac{d^2y}{dx^2} \right)^2 dx}$$

Energy Due to Axial Load (Stretching)



⇒ energy related to lengthening

$$ds = \left[(dx^2) + (dy)^2 \right]^{\frac{1}{2}} = dx \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}}$$

Binomial Theorem

$$\approx dx \left[1 + \frac{1}{2} \left(\frac{dy}{dx} \right)^2 \right]$$

$$\therefore \epsilon_x = \frac{ds - dx}{dx} = \frac{1}{2} \left(\frac{dy}{dx} \right)^2$$

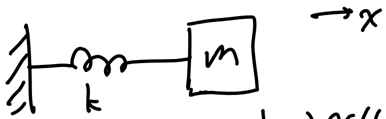
$$\left[dW_{axial} = S \epsilon_x dx = \frac{1}{2} S \left(\frac{dy}{dx} \right)^2 dx \right]$$

$$\boxed{W_{axial} = \frac{1}{2} S \int_0^L \left(\frac{dy}{dx} \right)^2 dx}$$

Axial Strain Energy

⇒ look @ shear strain energy in your module

$\boxed{\text{Estimating Resonance Frequency}}$



$\mapsto x(t) = X_o \cos\omega t$

Potential Energy

$$W(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kX_o^2\cos^2\omega t$$

amplitude $= X_o$

Kinetic Energy

$$K(t) = \frac{1}{2}M\dot{x}^2(t) = \frac{1}{2}MX_o^2\omega^2\sin^2\omega t$$

$\dot{x} = \frac{dx}{dt} =$ velocity

velocity $= \omega X_o$

Remarks:

① Energy must be conserved.

② Total Energy = Potential Energy + Kinetic Energy

at all times & locations in the structure!

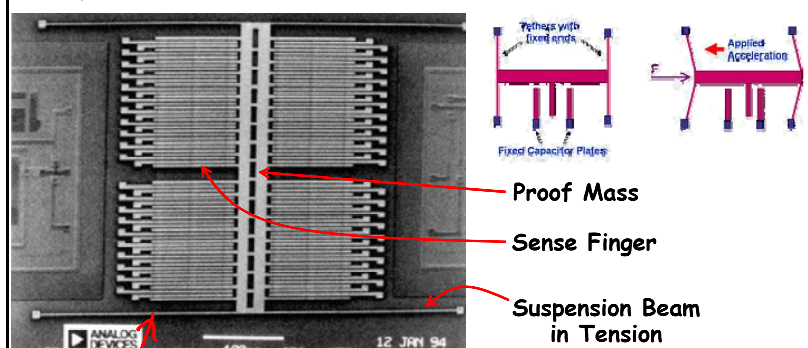
$$W_{max} = \frac{1}{2}kX_o^2 = K_{max} = \frac{1}{2}M\omega^2X_o^2$$

maximum potential energy — peak displacement — radian freq. — peak velocity

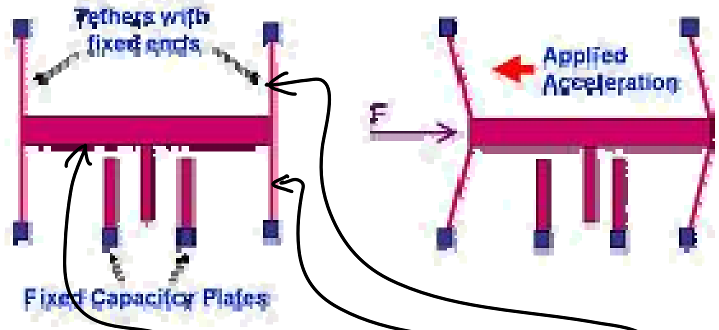
$$\therefore \quad \boxed{\omega = \sqrt{\frac{k}{M}}}$$

⇒ good for problems where mass & stiffness can be separated..

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



Suspension beam → when fabricated, purposely introduce a tension in the beams!



mass of structure (shuttle) >> mass of springs

∴ ignore the mass of the springs

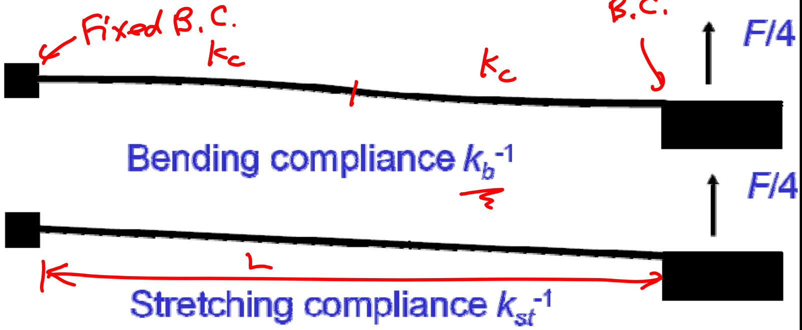
stiffness of the springs << stiffness of shuttle

∴ ignore the stiffness of the shuttle

For the ADXL50: $M = 162\,ng$

← 60% of this is in the fingers!

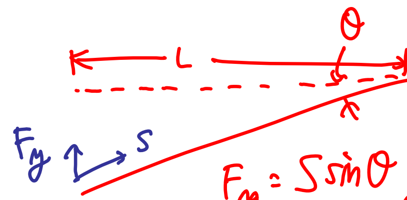
Suspension: four tensioned beams



Bending Contribution

$$k_b^{-1} = \left(\frac{1}{k_c}+\frac{1}{k_c}\right) = 2\left[\frac{(L/2)^3}{3E(Wh^3/12)}\right] = \frac{L^3}{EWh^3} = 4.2\,\mu m/\mu N$$

Stretching Contribution



$$F_y = S\sin\theta \approx S\theta \approx S\left(\frac{y}{L}\right) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}} y$$

[assume small displacement]

$$k_{st}^{-1} = \frac{L}{S} = \frac{L}{\sigma_r W h} = 1.14\,\mu m/\mu N$$

stretching stiffness

To get the total spring constant add the bending stiffness to the stretching:

$$k = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.5\,\mu N/\mu m$$

Now, get the resonance freq.:

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.5\,N/m}{162\times10^{-12}\,kg}} = 26.5\,kHz$$

ADXL-50 Data Sheet: $f_o = 24\,kHz$ ← difference?

capacitive transducer
↳ shifts freq.

introduces a negative
stiffness, k_e

↑ electrical stiffness!