Lecture 19: Resonance Frequency

- Announcements:
- Reminder: Midterm will be Tuesday, Nov. 2
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- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
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- Last Time:

$$W_{max} = \frac{1}{2}kX_0^2 = K_{max} = \frac{1}{2}M\omega^2X_0^2$$

maximum potential energy

peak displacement

radian freq.

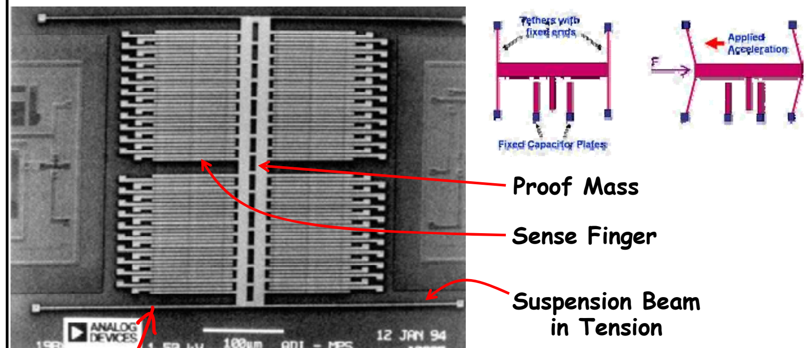
peak velocity

radian → $\times\frac{1}{2\pi}$ → Hz

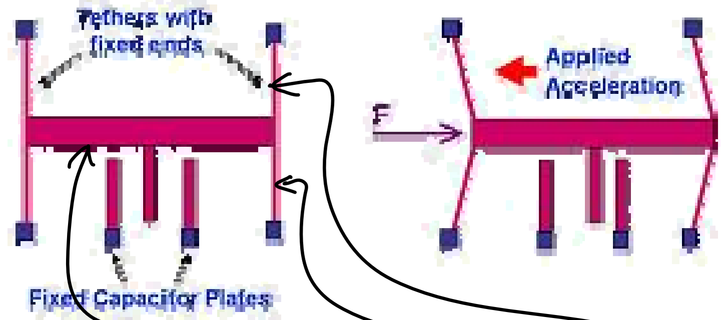
$$\therefore \boxed{\omega = \sqrt{\frac{k}{M}}}$$

⇒ good for problems whose mass & stiffness can be separated..

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



Suspension beam → when fabricated, purposely introduce a tension in the beams!



mass of structure (shuttle) >> mass of springs

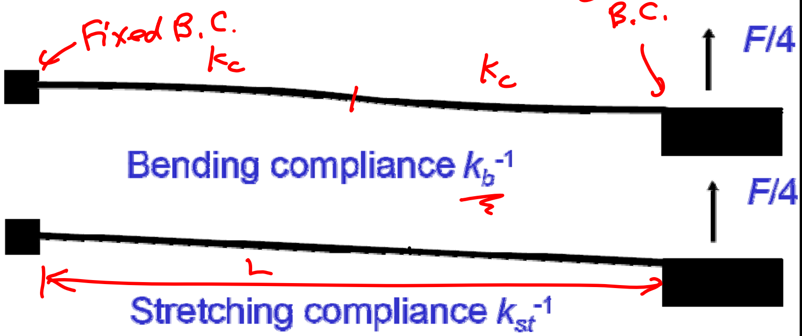
∴ ignore the mass of the springs

stiffness of the springs << stiffness of shuttle

∴ ignore the stiffness of the shuttle

For the ADXL50: $M = 162\,ng$
← 60% of this is in the fingers!

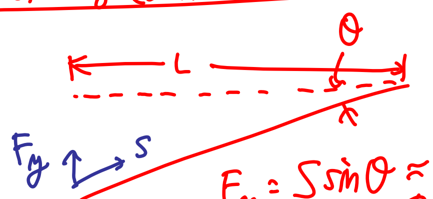
Suspension: four tensioned beams



Bending Contribution

$$k_b^{-1} = \left(\frac{1}{k_c} + \frac{1}{k_c}\right) = 2\left[\frac{(L/2)^3}{3E(Wh^3/12)}\right] = \frac{L^3}{EWh^3} = 4.2\,\mu m/\mu N$$

Stretching Contribution



$$F_y = S\sin\theta \approx S\theta \approx S\left(\frac{y}{L}\right) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}} y$$

≈ tanθ

[assume small displacement]

stretching stiffness

$$k_{st}^{-1} = \frac{L}{S} = \frac{L}{\sigma_r Wh} = 1.14\,\mu m/\mu N$$

To get the total spring constant
→ add the bending stiffness to the stretching:

$$k = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.5\,\mu N/\mu m$$

Now, get the resonance freq.:

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.5\,N/m}{162\times10^{-12}\,kg}} = 26.5\,kHz$$

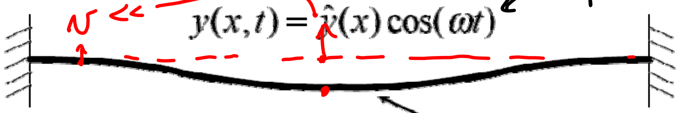
ADXL-50 Data Sheet: $f_o = 24\,kHz$ ↔ difference?

capacitive transducer
→ shifts freq.
→ introduces a negative stiffness, k_e
↑ electrical stiffness!

Resonance Frequency for Distributed Structures

- Vibrating structure displacement function:

v << $y(x,t) = \hat{y}(x)\cos(\omega t)$ ← displacement



Maximum displacement function (i.e., mode shape function) $\hat{y}(x)$
Seen when velocity $\dot{y}(x,t) = 0$

- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when t=0, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

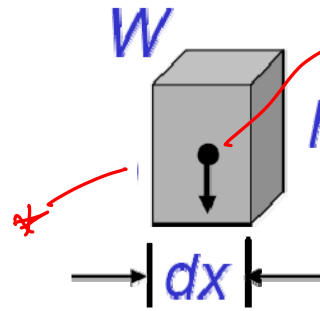
Get Maximum Kinetic Energy

velocity: $v(x,t) = \frac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin\omega t$ *

Velocity topographical mapping

when $y(x,t)=0$, all energy in the structure is kinetic: (since $W=0$)

$$v\left(x, \frac{(2n+1)\pi}{(2\omega)}\right) = -\omega\hat{y}(x) \rightarrow t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \dots$$



velocity:

$v = -\omega\hat{y}(x)$

$dK = \frac{1}{2} \cdot dm \cdot [v(x,t)]^2$

$dm = \rho(Wh\,dx)$

∴ Maximum Kinetic Energy:

$$K_{max} = \int_0^L \frac{1}{2}\rho W h\,dx\,v^2(x,t) = \int_0^L \frac{1}{2}\rho W h \omega^2 \hat{y}^2(x)\,dx$$

To get frequency:

$K_{max} = W_{max}$

$$\therefore \boxed{\omega = \sqrt{\frac{W_{max}}{\int_0^L \frac{1}{2}\rho W h \hat{y}^2(x)\,dx}}} \quad [\text{radians/s}]$$

Rayleigh-Ritz Method

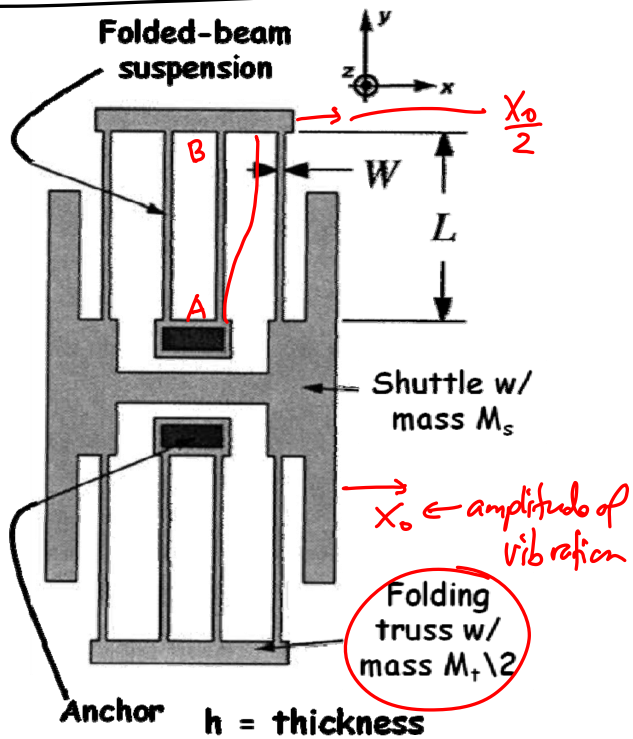
ω = res. freq.
W_{max} = max. potential energy
ρ = density of the structural material

W = beam width

h = thickness

$\xi(x)$ = resonance mode shape

$\boxed{\text{Determine } f_o \text{ of a Folded-Beam Resonator}}$



- Derive an expression for the resonance frequency of the above structure

Use the Rayleigh-Ritz method:

$$KE_{max} = PE_{max} \leftarrow W_{max}$$

Kinetic Energy:

$$KE_{max} = \underbrace{KE_s}_{\text{shuttle}} + \underbrace{KE_t}_{\text{truss}} + \underbrace{KE_b}_{\text{beams}}$$

$$= \frac{1}{2}v_s^2 M_s + \frac{1}{2}v_t^2 M_t + \frac{1}{2}\int v_b^2 dM_b$$

Velocity of Shuttle: $v_s = \omega_o X_o$

resonance freq. — maximum displacement of the shuttle

$$\therefore KE_s = \frac{1}{2}v_s^2 M_s = \frac{1}{2}\omega_o^2 X_o^2 M_s$$

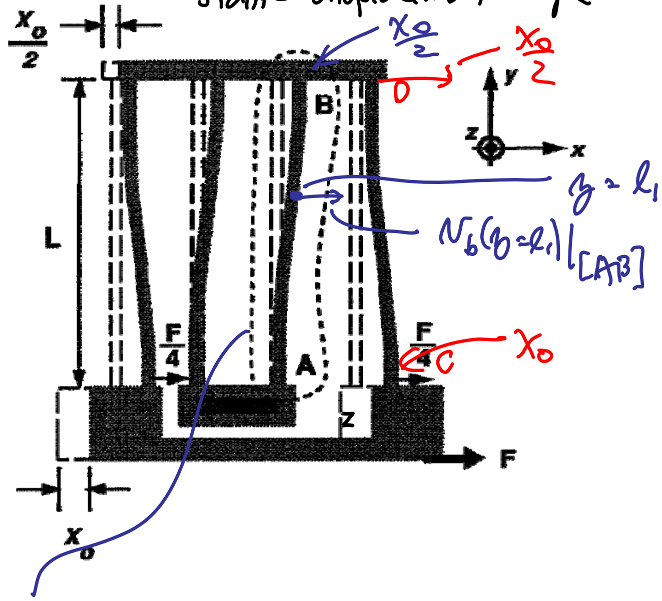
Velocity of the Truss: $v_t = \frac{1}{2}v_s = \frac{1}{2}\omega_o X_o$

$$\therefore KE_t = \frac{1}{2}\left(\frac{1}{2}\omega_o X_o\right)^2 M_t = \frac{1}{8}\omega_o^2 X_o^2 M_t$$

mass of both trusses together

Velocity of the Beam Segments:

⇒ assume the mode shape is the same as the static displacement shape



Segment AB:

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right) \qquad 0 \le y \le L \quad (1)$$

At $y = L$: $x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48EI_z}$ ← B.C.

Substitute into (1):

$$\hat{x}(y) = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\big|_{[AB]} = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_0$$

Plugging into the expression for KE_b:

$$KE_{[AB]} = \frac{1}{2}\int_0^L \frac{X_0^2\omega_0^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$$= \frac{X_0^2\omega_0^2 M_{[AB]}}{8L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

Static Mass of beam [AB]

$$KE_{[AB]} = \frac{13}{280}X_0^2\omega_0^2 M_{[AB]}$$

For Segment CD:

$$v_b(y)\big|_{[CD]} = X_0\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_0$$

Thus,

$$KE_{[CD]} = \frac{X_0^2\omega_0^2 M_{[CD]}}{2L}\int_0^L \left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]^2 dy$$

$$KE_{[CD]} = \frac{83}{280}X_0^2\omega_0^2 M_{[CD]}$$

static mass of beam [CD]

Let $M_b \triangleq$ total mass of all 8 beams

Then: $M_{[AB]} = M_{[CD]} = \frac{1}{8} M_b$

Thus:

$$KE_b = 4KE_{[AB]} + 4KE_{[CD]} = \frac{6}{35} X_o^2 \omega_o^2 M_b$$

and

$$KE_{max} = X_o^2 \omega_o^2 \left[\frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b\right]$$

$PE_{max} \rightarrow$ simply the work done to achieve max deflection

$$PE_{max} = \frac{1}{2} k_x X_o^2$$

Then, using Rayleigh-Ritz:

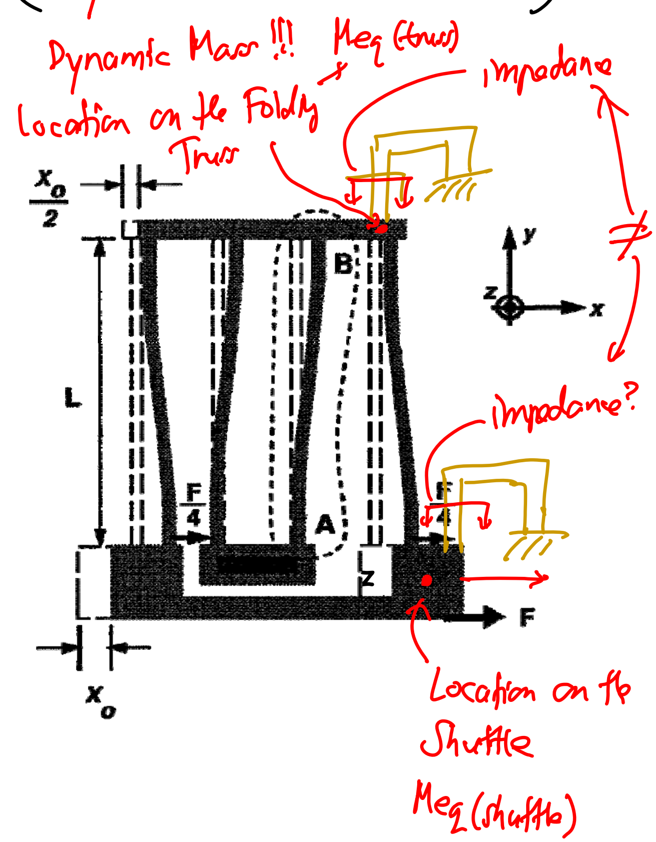
$$KE_{max} = PE_{max}$$

$$\cancel{X_o^2} \omega_o^2 \left[\frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b\right] = \frac{1}{2} \overset{k_c}{k_x} \cancel{X_o^2}$$

- Go through Module 10 slides 21-31 on your own

$$\boxed{\omega_o = \left[\frac{k_c}{M_{eq}}\right]^{1/2} \quad \text{where } M_{eq} = M_s + \frac{1}{4} M_t + \frac{12}{35} M_b}$$

(Resonance Freq. of a Folded-Beam Suspended Shuttle)



Equivalent Mass:

$$\text{Equiv. Mass: } M_{eq_x} = \frac{KE_{max}}{\frac{1}{2}V_x^2} = \frac{\frac{1}{2}\rho A\int_0^l V_x^2(x)\,dx}{\frac{1}{2}V_x^2}$$

velocity in x direction @ the location in question