Lecture 21: Capacitive Transducers

- **Announcements:**
- HW#6 is online and due Tuesday, Nov. 23
- Exams passed back today
- No lecture on Thursday, 11/11/10 (holiday)
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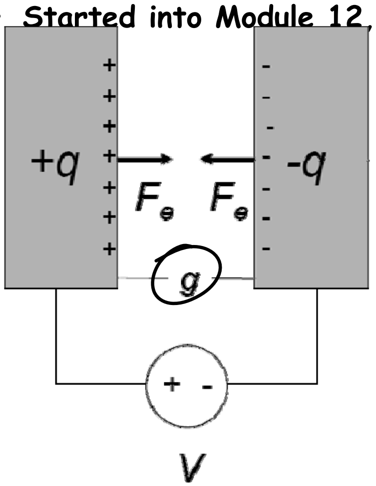
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- ----------------------------------
- **Last Time:** Energy Conserving Transducers
- Started into Module 12, slide 3



Goal: Determine gap spacing g as a function of input variables: V, I

q

F_e

g

Notes: Assume the plates are supported elastically

How do we proceed? → ① Determine the energy of the system.

② What can I Δ to Δ the energy in the system?

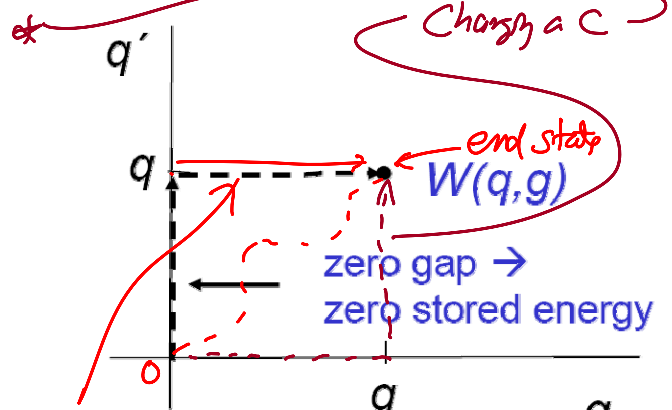
(i) change the charge, q

(ii) change the separation, g

$$\Delta W(q,g) = V\Delta q + F_e \Delta g$$

$$dW = V dq + F_e dg \leftarrow$$

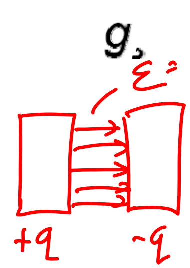
$\boxed{\text{Stored Energy}}$



No change in charge: $dq = 0$

$$W = 0 + \int_0^g F_e \, dg'$$

$$F_e = \left(\frac{q}{2}\right)\mathcal{E} = \frac{1}{2}\frac{q^2}{\epsilon A}$$



$\mathcal{E} = \frac{q}{\epsilon A}$

↑ cross-sectional area

$\therefore W = \int_0^g F_e dg' = F_e g'\Big|_0^g = F_e g$

$$\boxed{W(g) = \frac{1}{2}\frac{q^2}{\epsilon A} g}$$

* Work done to charge a C to q at fixed gap:

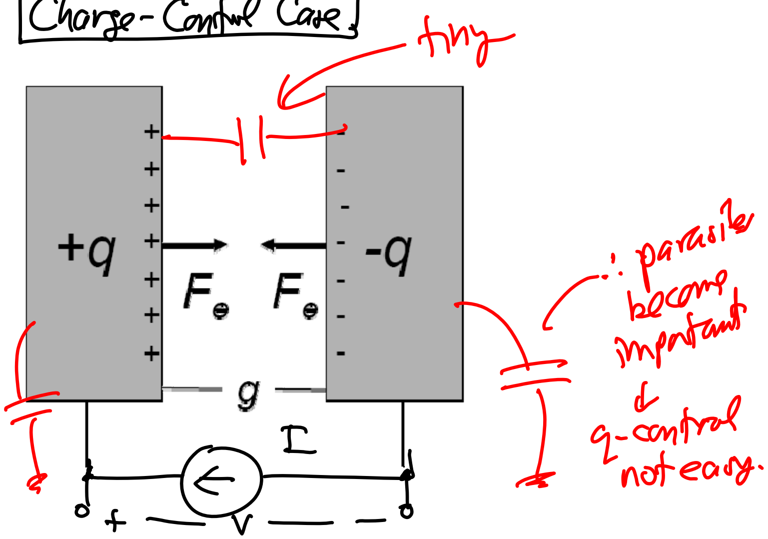
For a capacitor:

$q = CV \rightarrow V = \frac{q}{C}$

$\therefore W(q) = \int_0^q V dq = \int_0^q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{q^2}{C}$

$$= \boxed{\frac{1}{2}\frac{q^2}{\epsilon A} g = W(q)}$$

Charge-Control Case



From $dW = V\cancel{dq} + F_e dg$ — 0 if q = const.

⇒ Force is given by:

$$F_e = \frac{\partial W(q,g)}{\partial g}\Big|_{q=const.} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A} g\right)$$

$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}}$ ⇒ independent of gap spacing!

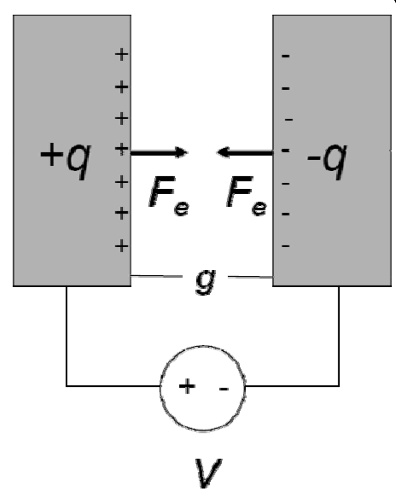
⇒ Voltage is given by:

$$V = \frac{\partial W(q,g)}{\partial q}\Big|_{g=const.} = \frac{\partial}{\partial q}\left(\frac{1}{2}\frac{q^2}{\epsilon A} g\right)$$

$$= \frac{qg}{\epsilon A} \quad \therefore \boxed{V = \frac{q}{C}} \checkmark$$

(consistent w/ what we already know)

Voltage Control



Want to write:

$F_e = f(V)$

We know this:

$dW = V dq + F_e dg$

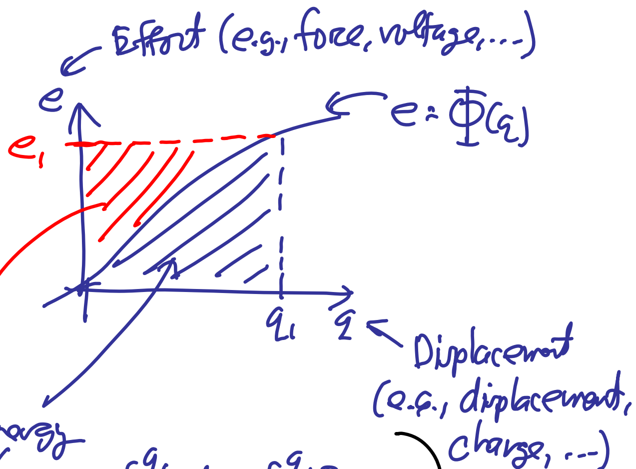
$W = W(q, g)$

Need: $W'(V,g)$

↑ replace charge q w/ V

Can get this by doing a Legendre transformation.

Energy & Co-Energy



Energy

$$W(q_1) = \int_0^{q_1} e\,dq = \int_0^{q_1} \Phi(q)\,dq$$

Co-Energy

$$W'(e_1) = \int_0^{e_1} q\,de = \int_0^{e_1} \Phi^{-1}(e)\,de$$

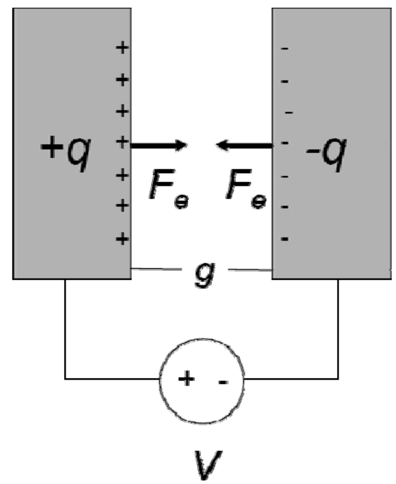
For a linear system, these will be equal.

Can define co-energy as:

$$W'(e) = eq - W(q) \quad \text{(from the plot)} \quad *$$

co-energy   energy

Co-Energy Formulation for Voltage-Control



$$W'(V,g) = qV - W(q,g)$$

* Differentially, this becomes:

$$dW'(V,g) = (q\,dV + \cancel{V\,dq}) - dW(q,g)$$

$$[dW(q,g) = F_e\,dg + \cancel{V\,dq}]$$

$$\boxed{dW'(V,g) = q\,dV - F_e\,dg}$$ ← Working Co-Energy Expression

*

Find co-energy in terms of voltage, V:

$$W' = \int_0^V q(g,V')\,dV' = \int_0^V \left(\frac{\epsilon A}{g}\right) V'\,dV'$$

$$= \frac{1}{2}\left(\frac{\epsilon A}{g}\right)V^2 = \frac{1}{2}CV^2 \quad \text{(as expected)}$$

Electrostatic (or Voltage-Controlled) Force:

$$F_e = -\left.\frac{\partial W'(V,g)}{\partial g}\right|_{V=const.}$$

$$= -\frac{1}{2}\left(-\frac{\epsilon A}{g^2}\right)V^2 = \boxed{\frac{1}{2}\frac{C}{g}V^2 = F_e}$$

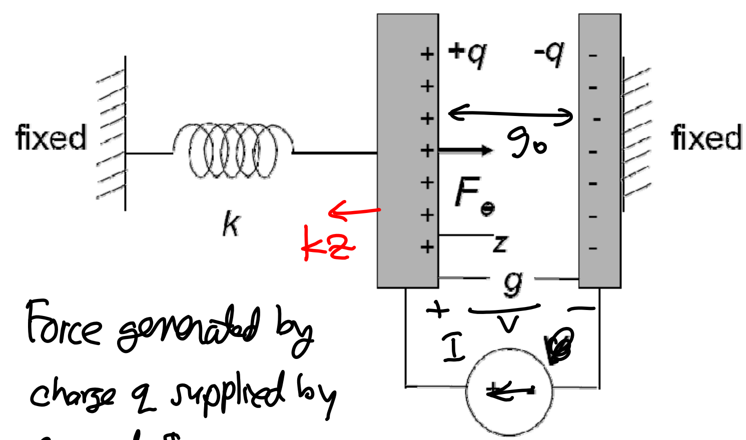
depends on gap!

Charge:

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_{g=const.} = \frac{\epsilon A}{g}V = CV$$

(as expected)

Charge-Control of a Spring-Suspended C



Force generated by charge q supplied by current I:

$$F_e = \left.\frac{\partial W(q,g)}{\partial g}\right|_q = \frac{q^2}{2\epsilon A}$$

Restoring force of spring: $F_{spring} = kz = F_e$

equilibrium

The gap, g:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

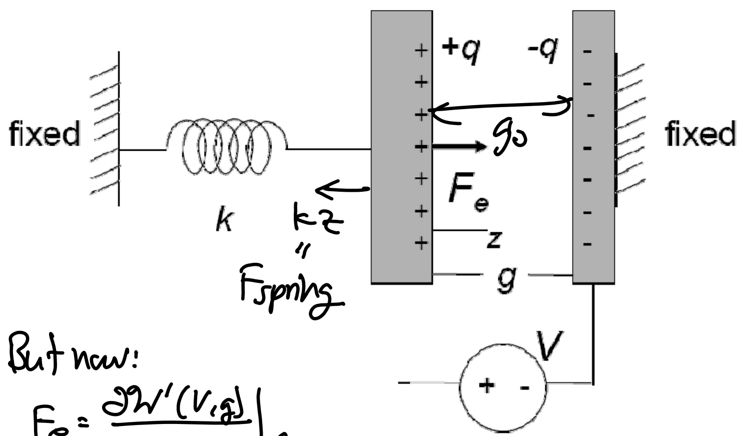
initial gap

$q\uparrow$ can drive $g \to 0$ in a continuous fashion

$$V = \frac{q}{C} = \frac{q}{\epsilon A}g = \boxed{\frac{q}{\epsilon A}\left(g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V}$$

$V\downarrow$ as $g\downarrow$

Voltage-Control of a suspended C



But now:

$$F_e = \left.\frac{\partial W'(V,g)}{\partial g}\right|_q$$

$$F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

↑ initial gap spacing

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1, then this will go unstable!

- Go through the midterm solutions and handout midterms