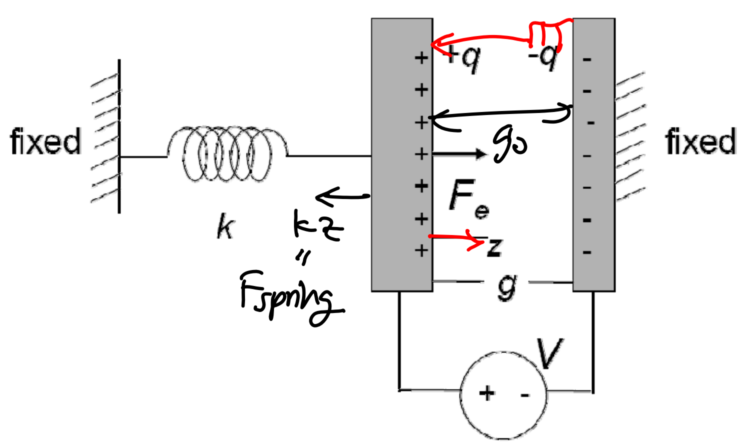
Lecture 22: Electrical Stiffness

- **Announcements:**
- None
- ------------------------------------
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- ------------------------------------
- **Last Time:**

$\boxed{\text{Voltage-Control of a spring-suspended C}}$



But now:

$$F_e = \left.\frac{\partial W'(V,g)}{\partial g}\right|_q$$

$$F_e = \frac{1}{2}\frac{\varepsilon A}{g^2}V^2 \leftarrow \text{non linear in } V \rightarrow \text{square law}$$

And the gap:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{\varepsilon A}{g^2}\frac{V^2}{k} = g}$$

↑ initial gap spacing

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$ → (+) Feedback!

If loop gain > 1, then this will go unstable!

plates will collapse into one another!

Charge: (for a variable gap)

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_g = CV \checkmark$$

Stability Analysis

→ determine under what conditions voltage-control causes collapse of the plates ...

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_0 - g)}_{F_{spring}}$$

What happens when I change g by an increment d_g?
↳ get an increment in net attractive force dF_{net}

$$\underset{(-)}{dF_{net}} = \frac{\partial F_{net}}{\partial g} d_g = \left[-\frac{\epsilon A V^2}{g^3} + k \right] \underset{(-)}{d_g}$$

If $g\downarrow \to d_g = (-)$, then for stability need:

$F_{net}\downarrow \to dF_{net} = (-)$

This needs to be (+)! → otherwise, the plates collapse into one another

Thus: $\boxed{k > \frac{\epsilon A V^2}{g^3}}$ (for a stable uncollapsed state)

Pull-in Voltage & Gap

$V_{PI} \triangleq$ voltage @ which plates collapse

$g_{PI} \triangleq$ gap spacing @ which " "

The plate goes unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \quad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_0 - g_{PI}) \quad (2)$$

pull-in voltage

pull-in gap

Substituting (1) into (2):

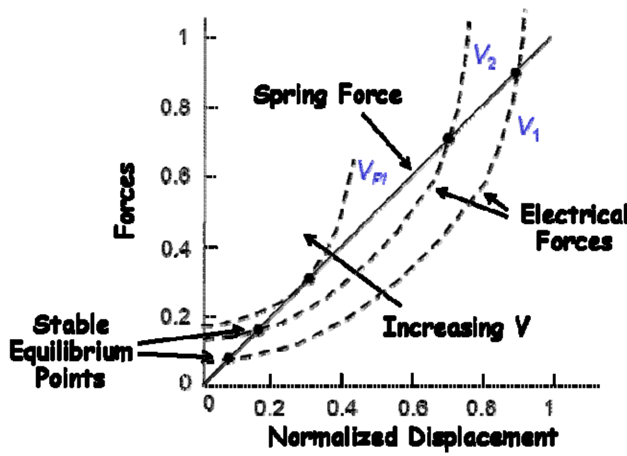
$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2 g_{PI}^{\cancel{2}}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_0 - g_{PI})$$

$$\frac{g_0 - g_{PI}}{g_{PI}} = \frac{1}{2} \to g_0 = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_0}$$

When the gap is driven by a voltage to (2/3) the initial spacing → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow \boxed{V_{PI} = \sqrt{\frac{8}{27}\frac{k g_0^3}{\epsilon A}}}$$



Advantages of Electrostatic Actuators:

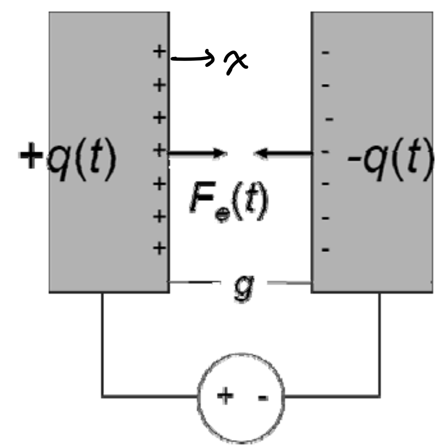
- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale

- Go through variable naming convention in slide 21 of Lecture Module 12

$\boxed{\text{Linearizing the Voltage to Force Transfer Fcn.}}$



$$v(t) = \underbrace{V_P}_{\text{Bias (DC)}} + \underbrace{v_i(t)}_{\text{Signal (AC)}}$$

$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}C\,(v(t))^2\right]$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}[v(t)]^2 = \frac{1}{2}\frac{\partial C}{\partial x}[V_P + v_i(t)]^2$$

$$= \frac{1}{2}\left[V_P^2 + 2V_P v_i(t) + \cancel{[v_i(t)]^2}\right]\frac{\partial C}{\partial x}$$

$$[V_P \gg v_i(t)] \Rightarrow F_e(t) = \underbrace{\frac{1}{2}V_P^2\frac{\partial C}{\partial x}}_{\text{DC offset}} + \underbrace{V_P\frac{\partial C}{\partial x}v_i(t)}_{\text{AC drive signal}}$$

$$\left[C_o = \frac{\epsilon A}{g_o}\right] \Rightarrow C(x) = \frac{\epsilon A}{g_o - x} = C_o\left(1 - \frac{x}{g_o}\right)^{-1}$$

$$[x \ll g_o] \Rightarrow C(x) \approx C_o\left(1 + \frac{x}{g_o}\right)$$

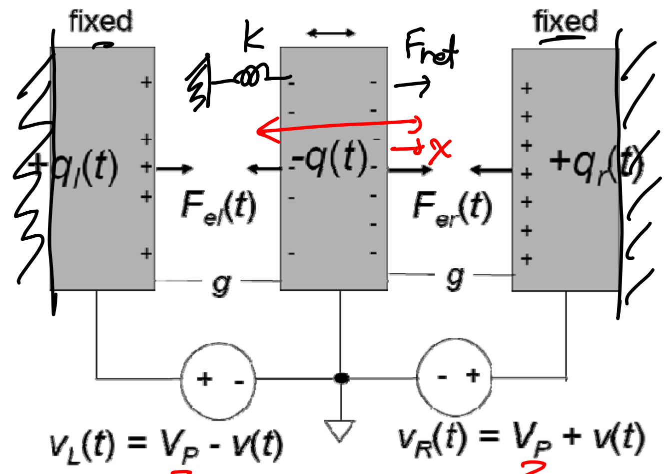
(using the Binomial Theorem)

$$\therefore \frac{\partial C}{\partial x} = \frac{C_o}{g_o}$$

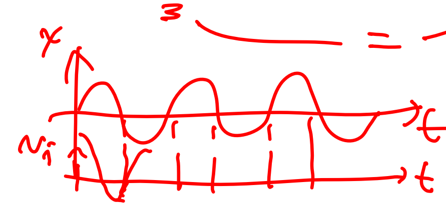
$$\Rightarrow F_e(t) = \underbrace{\frac{1}{2}\frac{C_o}{g_o}V_P^2}_{\text{Can cancel by Symmetry}} + \underbrace{V_P\frac{C_o}{g_o}}_{\sim\text{const. for small amplitudes}}v_i(t)$$

linear!

This is small-signal analysis in the mechanical domain!



$v_L(t) = V_P - v(t)$ $\quad$ $v_R(t) = V_P + v(t)$



$$F_{net}(t) = F_{er}(t) - F_{el}(t)$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{[v_R(t)]^2 - [v_L(t)]^2\right\}$$

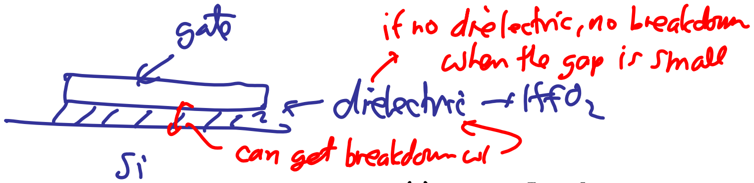
$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{\cancel{V_P^2} + 2V_P v(t) + \cancel{[v(t)]^2} - \left(\cancel{V_P^2} - 2V_P v(t) + \cancel{[v(t)]^2}\right)\right\}$$

$$\boxed{F_{net}(t) = 2V_P\frac{\partial C}{\partial x}v(t) = 2V_P\frac{C_o}{g_o}v(t)}$$

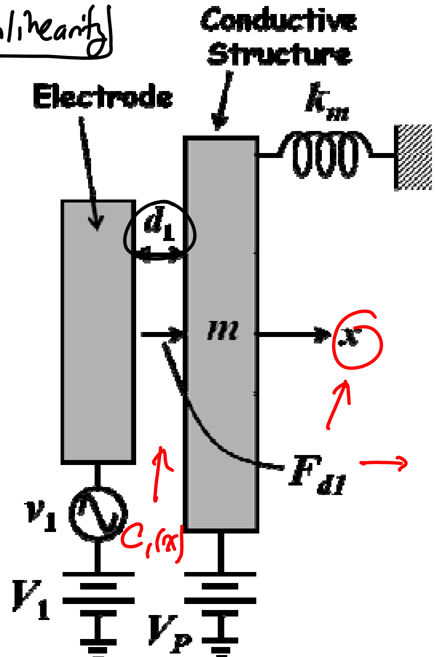
$$\frac{\epsilon A}{g_o^2}$$

$V_P\uparrow \to$ amplifier for force

$g_o\downarrow \to$ really " " "



Other Consequences of Nonlinearity



More Complete Expression

$$C_1(x) = \frac{\epsilon A}{d_1+x} = C_0\left(1+\frac{x}{d_1}\right)^{-1} \rightarrow \frac{\partial C_1}{\partial x} = -\frac{C_{01}}{d_1}\left(1+\frac{x}{d_1}\right)^{-2}$$

[Expand the Taylor series further]

$$\frac{\partial C_1}{\partial x} = -\frac{C_{01}}{d_1}\left(1+A_1x+A_2x^2+A_3x^3+\cdots\right)$$

where $A_1 = -\frac{2}{d_1}$, $A_2 = \frac{3}{d_1^2}$, $A_3 = -\frac{4}{d_1^3}$, …

$$F_{d1} = \frac{1}{2}\frac{\partial C_1}{\partial x}\left(V_P - V_1 - v_1\right)^2 = \frac{1}{2}\frac{\partial C_1}{\partial x}\left(V_{P1} - v_1\right)^2$$

[small displacement: $x \ll d_1$] $V_{P1} = V_P - V_1$

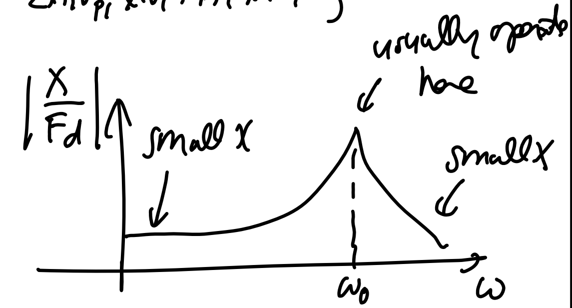
$$F_{d1} = \frac{1}{2}\left(-\frac{C_{01}}{d_1}\right)(1+A_1x)\left(V_{P1}^2 - 2V_{P1}v_1 + v_1^2\right)$$

@ ω_o → $v_1 = V_1 \cos\omega_o t$

$$= \frac{1}{2}\left(-\frac{C_{01}}{d_1}\right)\left\{V_{P1}^2 - 2V_{P1}v_1 + v_1^2 + A_1V_{P1}^2x - 2A_1V_{P1}xv_1 + A_1xv_1^2\right\}$$

@ ω_o

Resonance:



@ resonance:

$$x = \frac{QF_{d1}}{jk} = \frac{Q}{jk}\frac{\partial C}{\partial x}V_{P1}v_1$$

90° phase shift → displacement x is 90° phase-shifted from force, F_d

Thus:

$v_1 = |v_1|\cos\omega_o t \rightarrow x = |X|\sin\omega_o t$

phase-shifted 90°

Force terms only @ ω_o:

$$F_{d1}\big|_{\omega_o} = \underbrace{V_{P1}\frac{C_{o1}}{d_1}|v_i|\cos\omega_o t}_{\text{forcing term}} + \overbrace{V_{P1}^2\frac{C_{o1}}{d_1^2}}^{k_e \to \text{electrical stiffness}}|x|\sin\omega_o t$$

proportional to x

90°

in phase w/ displacement

Electrical stiffness:

① A negative spring constant!

② Derives from V_P:

$$\boxed{k_e = V_{P1}^2\frac{C_{o1}}{d_1^2} = V_{P1}^2\frac{\epsilon A}{d_1^3}}$$