


EE C245 - ME C218
Introduction to MEMS Design
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Prof. Clark T.-C. Nguyen

Dept. of Electrical Engineering & Computer Sciences
University of California at Berkeley
Berkeley, CA 94720

Lecture Module 13: Equivalent Circuits II

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Lecture Outline

- Reading: Senturia, Chpt. 6, Chpt. 14
- Lecture Topics:
 - ↗ Input Modeling
 - Force-to-Velocity Equiv. Ckt.
 - Input Equivalent Ckt.
 - ↗ Current Modeling
 - Output Current Into Ground
 - Input Current
 - Complete Electrical-Port Equiv. Ckt.
 - ↗ Impedance & Transfer Functions

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Input Modeling

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Electromechanical Analogies

$F(t) = F \cos(\omega t) \rightarrow x(t) = X \cos \omega t$
Equation of Motion:
 $m_{eq} \ddot{x} + c_{eq} \dot{x} + k_{eq} x = F(t)$
 \Rightarrow using phasor concepts:
 $F = j\omega m_{eq} \dot{x} + \frac{k_{eq}}{j\omega} x + c_{eq} \dot{x}$
 \Rightarrow by analogy:

$F \rightarrow N$	$m_{eq} \rightarrow l_x$	$c_{eq} \rightarrow r_x$
$\dot{x} \rightarrow i$	$k_{eq} \rightarrow \frac{1}{c_x}$	

 [Parameter Relationships in the Current Analogy]

$N(t) = V \cos \omega t \rightarrow i(t) = I \cos \omega t$
Impedance looking in:
 $\frac{N}{i} = j\omega l_x + \frac{1}{j\omega c_x} + r_x$
 $N = j\omega l_x i + \frac{(1/c_x)}{j\omega} i + r_x i$

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Bandpass Biquad Transfer Function

$F = j\omega m_{eq} \dot{x} + \frac{k_{eq}}{j\omega} x + C_{eq} \dot{x}$
 \Rightarrow Converting to full phasor form:
 $F = (j\omega)(j\omega X) m_{eq} + \frac{k_{eq}}{j\omega} (j\omega X) + C_{eq} (j\omega X)$
 $\frac{X}{F}(j\omega) = \frac{1}{k_{eq}} \left[-\omega^2 \frac{m_{eq}}{k_{eq}} + 1 + j \frac{C_{eq}\omega}{k_{eq}} \right]^{-1} = \frac{1}{k_{eq}} \left[-\left(\frac{\omega}{\omega_0}\right)^2 + 1 + j \frac{\omega}{Q\omega_0} \right]^{-1}$
 $\left[\frac{k_{eq}}{m_{eq}} = \omega_0^2, Q = \frac{m_{eq}\omega_0}{C_{eq}} = \frac{k_{eq}}{\omega_0 C_{eq}} \rightarrow \frac{k_{eq}}{C_{eq}} = Q\omega_0 \right]$

$$\frac{X}{F}(j\omega) = \frac{k_{eq}^{-1}}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j \frac{\omega}{Q\omega_0}}$$

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Force-to-Velocity Relationship

- The relationship between input voltage v_1 and force F_{d1} :

$$F_{d1} \approx -V_P \frac{\partial C_1}{\partial x} v_1$$
- When displacement x is the mechanical output variable:

$$\frac{X(s)}{F_{d1}(s)} = \frac{1}{k} \frac{\omega_0^2}{s^2 + (\omega_0/Q)s + \omega_0^2}$$
- When velocity v is the mechanical output variable:

$$\frac{v(s)}{F_{d1}(s)} = \frac{sX(s)}{F_{d1}(s)} = \frac{1}{k} \frac{\omega_0^2 s}{s^2 + (\omega_0/Q)s + \omega_0^2}$$

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Force-to-Velocity Equiv. Ckt.

• Combine the previous lumped LCR mechanical equivalent circuit with a circuit modeling the capacitive transducer → circuit model for voltage-to-velocity

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Equiv. Circuit for a Linear Transducer

• A transducer ...

- ↔ converts energy from one domain (e.g., electrical) to another (e.g., mechanical)
- ↔ has at least two ports
- ↔ is not generally linear, but is virtually linear when operated with small signals (i.e., small displacements)

Electrical Mechanical

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Equiv. Circuit for a Linear Transducer

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Current $\rightarrow I$
 Voltage $\rightarrow V$

Linear Two-Port Element

$U = -\dot{x}$ Velocity
 Force F

Electrical | Mechanical

- For physical consistency, use a transformer equivalent circuit to model the energy conversion from the electrical domain to mechanical domain

$$\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\frac{1}{\eta} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix}$$

Describing Matrix

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Electromechanical Equivalent Circuit

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F_{d1}
 d_1
 Electrode 1
 i_1
 v_1
 V_P
 C_1
 m
 x
 b
 k

- $e_2 = F_{d1}$, $e_1 = v_1$, just need η_1 :
- From the matrix: $e_2 = \eta e_1$

$$F_{d1} \approx -V_P \frac{\partial C_1}{\partial x} v_1 \rightarrow \eta_1 = \left| V_P \frac{\partial C_1}{\partial x} \right|$$

Current $\rightarrow I_1$
 Voltage $\rightarrow V_1$

Velocity $\rightarrow U = -\dot{x}$
 Force F_{d1}

Electrical | Mechanical

$I_x = m$
 $r_x = b$
 $c_x = 1/k$

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Output Modeling

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Output Current Into Ground

- When the mass moves with time-dependent displacement $x(t)$, the electrode-to-mass capacitors $C_1(x,t)$ and $C_2(x,t)$ vary with time
- This generates an output current:

$$[q = CV] \Rightarrow i = \frac{dq}{dt} = C \frac{\partial V}{\partial t} + V \frac{\partial C}{\partial t}$$

$$i_2(t) = C_2(x,t) \frac{dV_2(t)}{dt} + V_2(t) \frac{dC_2(x,t)}{dt}$$

$$[V_2(t) = -V_P] \Rightarrow i_2 = -V_P \frac{dC_2}{dt} = -V_P \frac{\partial C_2}{\partial x} \frac{\partial x}{\partial t}$$

In phasor form: $I_2(j\omega) = -V_P \frac{\partial C_2}{\partial x} (j\omega X)$

$$I_2(j\omega) = -j\omega V_P \frac{\partial C_2}{\partial x} X$$

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Output Current Into Ground

$I_2(j\omega) = -j\omega V_p \frac{\partial C_2}{\partial x} X = -V_p \frac{\partial C}{\partial x} U$
 $90^\circ \text{ phase lag}$
 $(+)$ $(+)$ $\rightarrow I_2 = (-)$ when $x = (+)$ ✓

• Again, model with a transformer:

Velocity $\rightarrow U = \dot{x}$ Current $\rightarrow I_2$

$f_2 = -\frac{1}{\eta_2} f_1 \rightarrow f_1 = -\eta_2 f_2$
 $[f_1 = I_2, f_2 = U] \Rightarrow I_2 = -\eta_2 U$
 $\therefore \eta_2 = \left| V_p \frac{\partial C}{\partial x} \right|$

$I_2(j\omega) = -j\omega V_p \frac{\partial C_2}{\partial x} X = -V_p \frac{\partial C}{\partial x} U$
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 $\therefore \eta_2 = \left| V_p \frac{\partial C}{\partial x} \right|$

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Input Current Expression

Get $I_1(j\omega)$:

$$i_1(t) = C_1(x,t) \frac{dv_1(t)}{dt} + v_1(t) \frac{dC_1(x,t)}{dt}$$

$$[v_1(t) \cdot N_i - V_p] \Rightarrow i_1 = C_1 \frac{dv_1}{dt} + [M_i - V_p] \frac{\partial C_1}{\partial x} \frac{\partial x}{\partial t}$$

$$\therefore I_1(j\omega) = C_1(j\omega V_1) + V_1 \frac{\partial C_1}{\partial x} (j\omega X) - V_p \frac{\partial C_1}{\partial x} (j\omega X)$$

$$= j\omega C_1 V_1 + j\omega V_1 \frac{\partial C_1}{\partial x} X - j\omega V_p \frac{\partial C_1}{\partial x} X$$

$$[V_1 \ll V_p] \Rightarrow I_1(j\omega) = \underbrace{j\omega C_1 V_1}_{\text{Feed-through Current}} - \underbrace{j\omega V_p \frac{\partial C_1}{\partial x} X}_{\text{Motional Current (due to mass motion)}}$$

@ DC: $x = \frac{F_{dl}}{k} = -\frac{1}{k} V_p \left(\frac{\partial C_1}{\partial x} \right) V_1$

@ resonance: $x = \frac{Q F_{dl}}{jk} = -\frac{Q}{jk} V_p \frac{\partial C_1}{\partial x} V_1$
 $\hookrightarrow 90^\circ \text{ phase lag}$

Get $I_1(j\omega)$:

$$i_1(t) = C_1(x,t) \frac{dv_1(t)}{dt} + v_1(t) \frac{dC_1(x,t)}{dt}$$

$$[v_1(t) \cdot N_i - V_p] \Rightarrow i_1 = C_1 \frac{dv_1}{dt} + [M_i - V_p] \frac{\partial C_1}{\partial x} \frac{\partial x}{\partial t}$$

$$\therefore I_1(j\omega) = C_1(j\omega V_1) + V_1 \frac{\partial C_1}{\partial x} (j\omega X) - V_p \frac{\partial C_1}{\partial x} (j\omega X)$$

$$= j\omega C_1 V_1 + j\omega V_1 \frac{\partial C_1}{\partial x} X - j\omega V_p \frac{\partial C_1}{\partial x} X$$

$$[V_1 \ll V_p] \Rightarrow I_1(j\omega) = \underbrace{j\omega C_1 V_1}_{\text{Feed-through Current}} - \underbrace{j\omega V_p \frac{\partial C_1}{\partial x} X}_{\text{Motional Current (due to mass motion)}}$$

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 $\hookrightarrow 90^\circ \text{ phase lag}$

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Input Current Expression (cont)

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Thus: (@ resonance)

$$I_1(j\omega) = j\omega C_1 V_1 + j\omega_0 \left(V_P \frac{\partial C_1}{\partial x} \right)^2 \frac{Q}{jk} V_1$$

$$= j\omega_0 C_1 V_1 + \omega_0 \frac{Q}{k} \eta^2 V_1$$

90° phase-shifted from V_1 → This is a Capacitor in shunt w/ the input.
 In phase w/ V_1 → This is an effective resistance seen looking into Electrode 1

Motional Resistance:

$$R_{x1} \frac{V_1}{I_1} = \frac{k}{\omega_0 Q \eta^2} = \frac{m\omega_0}{Q \eta^2} = \frac{b}{\eta^2} = R_{x1}$$

(The equivalent ckt. better get this right!)

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Complete Electrical-Port Equiv. Circuit

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Static electrode-to-mass overlap capacitance

$l_x = m$
 $c_x = \frac{1}{k}$
 $r_x = b$

$\eta_{e1} = V_P \frac{\partial C_1}{\partial x} = V_P \frac{C_{o1}}{d_1}$
 $\eta_{e2} = V_P \frac{\partial C_2}{\partial x} = V_P \frac{C_{o2}}{d_2}$

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Input Impedance Into Port 1

• What is the impedance seen looking into port 1 with port 2 shorted to ground?

From our transformer model: $\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\frac{1}{\eta} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \Rightarrow e_2 = \eta e_1 \rightarrow e_1 = \frac{e_2}{\eta}$
 $f_2 = -\frac{1}{\eta} f_1 \rightarrow f_1 = -\eta f_2$

$\frac{e_1}{f_1} = \frac{e_2}{\eta} \left(\frac{1}{-\eta f_2} \right) = -\frac{1}{\eta^2} \frac{e_2}{f_2} \rightarrow \frac{V_1}{i_1} = z_i = -\frac{1}{\eta_{e1}^2} \frac{F_2}{f_2} = \frac{1}{\eta_{e1}^2} z_x$

$z_i = \frac{1}{\eta_{e1}^2} \left(j\omega L_x + \frac{1}{j\omega C_x} + r_x \right) = j\omega \underbrace{\left(\frac{L_x}{\eta_{e1}^2} \right)}_{L_{x1}} + \frac{1}{j\omega \underbrace{\left(\eta_{e1}^2 C_x \right)}_{C_{x1}}} + \frac{r_x}{\eta_{e1}^2} \underbrace{\phantom{\frac{1}{j\omega C_x}}}_{R_{x1}}$

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Input Impedance Into Port 2

• What is the impedance seen looking into port 2 with port 1 shorted to ground?

$\frac{V_2}{i_2} = z_i = \frac{1}{\eta_{e2}^2} \left(j\omega L_x + \frac{1}{j\omega C_x} + r_x \right) = j\omega \underbrace{\left(\frac{L_x}{\eta_{e2}^2} \right)}_{L_{x2}} + \frac{1}{j\omega \underbrace{\left(\eta_{e2}^2 C_x \right)}_{C_{x2}}} + \frac{r_x}{\eta_{e2}^2} \underbrace{\phantom{\frac{1}{j\omega C_x}}}_{R_{x2}}$

Note: These are not the same as L_{x1} , C_{x1} , & R_{x1} !

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Port 1 to 2 TransG Across the Circuit

• What is the transconductance from port 1 to port 2 with port 2 shorted to ground?

$$\dot{x} = \frac{1}{\eta_{e1}} \dot{i}_1$$

$$\dot{i}_o = \eta_{e2} \dot{x} \rightarrow \dot{i}_o = \frac{\eta_{e2}}{\eta_{e1}} \dot{i}_1 = \frac{\eta_{e2}}{\eta_{e1}} \left(\frac{N_1}{Z_1} \right) = \frac{\eta_{e2}}{\eta_{e1}} N_1 \left[\eta_{e1}^2 \frac{1}{j\omega L_x + \frac{1}{j\omega C_x} + r_x} \right]$$

$$\therefore \frac{\dot{i}_o}{N_1} (j\omega) = \frac{\eta_{e1}\eta_{e2}}{j\omega L_x + \frac{1}{j\omega C_x} + r_x} = \left[j\omega L_{x12} + \frac{1}{j\omega C_{x12}} + R_{x12} \right]^{-1} \begin{cases} L_{x12} = \frac{L_x}{\eta_{e1}\eta_{e2}} \\ C_{x12} = \eta_{e1}\eta_{e2}C_x \\ R_{x12} = \frac{r_x}{\eta_{e1}\eta_{e2}} \end{cases}$$

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Port 1 to 2 v_i -to- i_o Transfer Function

$$\frac{\dot{i}_o}{N_1} (j\omega) = \frac{\eta_{e1}\eta_{e2}}{j\omega L_x + \frac{1}{j\omega C_x} + r_x} = \left[j\omega L_{x12} + \frac{1}{j\omega C_{x12}} + R_{x12} \right]^{-1} \begin{cases} L_{x12} = \frac{L_x}{\eta_{e1}\eta_{e2}} \\ C_{x12} = \eta_{e1}\eta_{e2}C_x \\ R_{x12} = \frac{r_x}{\eta_{e1}\eta_{e2}} \end{cases}$$

Separate freq. response & magnitude:

$$\frac{\dot{i}_o}{N_1} (s) = \frac{1}{sL_x + \frac{1}{sC_x} + R_x} = \frac{sC_x}{s^2L_xC_x + 1 + sC_xR_x} = \frac{s(\frac{1}{L_x})}{s^2 + \frac{1}{L_xC_x} + s(\frac{R_x}{L_x})}$$

$$\left[\frac{1}{L_xC_x} = \omega_0^2, Q = \frac{\omega_0 L_x}{R_x} \rightarrow \frac{R_x}{L_x} = \frac{\omega_0}{Q} \right] \Rightarrow \frac{\dot{i}_o}{N_1} (s) = \frac{1}{R_x} \frac{s(\omega_0/Q)}{s^2 + s(\omega_0/Q) + \omega_0^2} = \frac{1}{R_x} \mathcal{H}(s)$$

$$\mathcal{H}(s) = \frac{s(\omega_0/Q)}{s^2 + s(\omega_0/Q) + \omega_0^2}$$

$s=0: \mathcal{H}(0) = 0$
 $s=j\omega_0: \mathcal{H}(j\omega_0) = 1$
 $s=\infty: \mathcal{H}(\infty) = 0$

This will always be the same. Thus, could just work @ resonance & just multiply by $\mathcal{H}(s)$.

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Condensed Equiv. Circuit (Symmetrical)

Holds for the symmetrical case, where port 1 and port 2 are identical

If $\eta_{e1} = \eta_{e2}$, then ...

where

$$L_x = \frac{m}{\eta_e^2}$$

$$C_x = \frac{\eta_e^2}{k}$$

$$R_x = \frac{b}{\eta_e^2}$$

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Phasings of Signals

- Below: plots of resonance electrical and mechanical signals vs. time, showing the phasings between them

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