EE290N Convex Optimization and Applications
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Final Project

H_∞ Optimization of PID Controller Parameters

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Abstract

In this project, the H_∞ optimization problem of a PID (Proportional-Integral-Derivative) controller parameters is considered. H_∞ optimization is the well-developed optimization method to synthesize the dynamics feedback controller that minimizes the specified input-output H_∞ norm of the closed-loop system. The H_∞ controller synthesis theories have enjoyed a great attention in control fields over years and its applicability to practical problems have been reported in many literatures. The standard algorithm usually gives, however, the optimal controller of the same order as the plant. Therefore, it cannot be directly applied to our problem.

Our algorithm starts with "extraction" of the controller parameters out of the closed-loop system as a full constant block by using the linear fractional transformations (LFTs). Then, optimization problem of the controller parameters can be seen as the H_∞-optimal static output feedback controller synthesis problem. It can be solved by using the LMI (Linear Matrix Inequality)-based H_∞ optimization algorithm, along with the cone complementarity linearization algorithm to overcome the non-convexity that arises due to the constraint on the controller order. As an application example, a two-inputs single-output PI controller is designed for steering control of lateral motion of heavy-duty vehicles (HDVs).

1 Problem Formulation

Suppose the closed-loop system is given as shown in Figure 1, where $P(s)$ and $C(s)$ represent a linear plant and controller, respectively. Here, the controller $C(s)$ a single-input single-output (SISO) PID controller with an approximated derivative action of the following form:

$$C(s) = -(k_p + k_i \frac{1}{s} + k_d \frac{s}{T_0 s + 1}) \tag{1}$$

where k_p, k_i and k_d are constant parameters that are independently tunable. T_0 is a small constant for the approximated derivative action and here it is assumed to be fixed. The controller dynamics can be also written in the following state space representation:

$$\begin{bmatrix} \dot{x}_i(t) \\ \dot{x}_d(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{T_0} \end{bmatrix} \begin{bmatrix} x_i(t) \\ x_d(t) \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{1}{T_0} \end{bmatrix} y(t) \tag{2}$$

$$u(t) = \begin{bmatrix} k_i & k_d(-\frac{1}{T_0}) \end{bmatrix} \begin{bmatrix} x_i(t) \\ x_f(t) \end{bmatrix} + \left[k_p + k_d(-\frac{1}{T_0})\right] y(t) \tag{3}$$

The optimization problem considered in this paper is given as follows:

$$\min_{k_1, \cdots, k_N} \|F_L(P(s), C(s))\|_\infty \tag{4}$$

where $F_L(P(s), C(s))$ denotes the closed-loop transfer function from d to z in Figure 1. $\| * \|_\infty$ denotes the induced L_2 norm from input signals (Here, d) to output signals (z). See Section 3 for a practical example of application of H_∞ optimization of the controller.

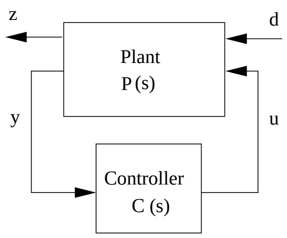

Figure 1: Standard closed-loop system configuration

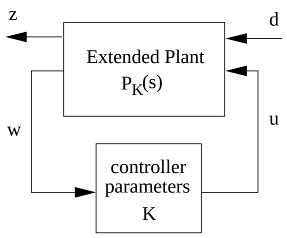

Figure 2: Extraction of controller parameters

2 H_∞ Optimization Algorithm of PID Controller Parameters

2.1 Step 1: Extraction of Controller Parameters

The proposed algorithm starts with "extraction" of controller parameters k_p, k_i and k_d out of the closed-loop system as a full constant block using the LFTs. Suppose the plant $P(s)$ is given in the following form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 d(t) + B_2 u(t) &(5)\\ z(t) &= C_1 x(t) + D_{11} d(t) + D_{12} u(t) &(6)\\ y(t) &= C_2 x(t) + D_{21} d(t) &(7)\end{aligned}$$

Then, define the extended plant $P_K(s)$ as follows:

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_i(t) \\ \dot{x}_d(t) \end{bmatrix} &= \begin{bmatrix} A & 0 & 0 \\ C_2 & 0 & 0 \\ -\frac{1}{T_0}C_2 & 0 & -\frac{1}{T_0} \end{bmatrix} \begin{bmatrix} x(t) \\ x_i(t) \\ x_d(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix} d(t) + \begin{bmatrix} B_2 \\ 0 \\ 0 \end{bmatrix} u(t) &(8)\\ z(t) &= \begin{bmatrix} C_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_i(t) \\ x_d(t) \end{bmatrix} + D_{11} d(t) + D_{12} u(t) &(9)\\ w(t) &= \begin{bmatrix} -C_2 & 0 & 0 \\ 0 & -1 & 0 \\ -\frac{1}{T_0}C_2 & 0 & -\frac{1}{T_0} \end{bmatrix} \begin{bmatrix} x(t) \\ x_i(t) \\ x_d(t) \end{bmatrix} + \begin{bmatrix} -D_{21} \\ 0 \\ 0 \end{bmatrix} d(t) &(10)\end{aligned}$$

Notice that $P_K(s)$ contains two controller state variables $x_i(s) = \frac{1}{s}y(s)$ and $x_d(s) = \frac{-1}{1+T_0 s}y(s)$. By using the outputs of this model, the control input $u(t)$ is given by

$$u(t) = \begin{bmatrix} k_p & k_d & k_i \end{bmatrix} w(t) = \begin{bmatrix} k_p & k_d & k_i \end{bmatrix} \begin{bmatrix} -y(t) \\ -x_i(t) \\ -\frac{1}{T_0}x_d(t) - \frac{1}{T_0}y(t) \end{bmatrix} \tag{11}$$

The entire closed-loop system can be seen as shown in Figure 2, where $K := \begin{bmatrix} k_p & k_d & k_i \end{bmatrix}$ is a full constant block.

This transformation implies that the H_∞ optimization problem of the controller parameters can be seen as the H_∞-optimal static output feedback controller synthesis problem for the plant P_K. The objective of application of this transformation is to avoid imposing constraints on the controller structure, except its order. The synthesis algorithm of the H_∞-optimal static output feedback controller will be presented in the next section.

Remark:

Clearly, application of this transformation is not restricted only to PID controllers. The optimization algorithm proposed in this paper can be applied to any linear controller structure that can be transformed into the form shown in Figure 2.

2.2 Step 2: H_∞ Optimization of Controller Parameters Block

Suppose the extended plant model $P_K(s)$ is given in the following form:

$$P_K = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right) \tag{12}$$

The plant dimensions are given by $A \in \Re^{n\times n}$, $D_{11} \in \Re^{p_1 \times m_1}$ and $D_{22} \in \Re^{p_2\times m_2}$. The optimization problem of the controller parameters can be now formulated as:

$$\min_{K\in\Re^{m_2\times p_2}} \|F_L(P_K(s), K)\|_\infty \tag{13}$$

As seen in [1], there exists a dynamic controller $K(s)$ of order k such that $\|F_L(P_K(s), K(s))\|_\infty < \gamma$ if and only if there exist two symmetric matrices R and S such that

$$\mathcal{N}_1^T \begin{bmatrix} AR + RA^T & RC_1^T & B_1 \\ C_1 R & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} \mathcal{N}_1 < 0 \tag{14}$$

$$\mathcal{N}_2^T \begin{bmatrix} A^T S + SA & SB_1^T & C_1^T \\ B_1^T S & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \mathcal{N}_2 < 0 \tag{15}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \tag{16}$$

$$rank\,(RS - I) \leq k \tag{17}$$

where $\mathcal{N}_1 = diag\,\{N_{12}, I\}\,, \mathcal{N}_2 = diag\,\{N_{21}, I\}$, N_{12} and N_{21} are bases of the null space of $\begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix}$ and $\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$, respectively.

Note that we must search for $K(s)$ of zeroth order, that is, we must search for R and S that satisfy (14)$\sim$(17) with $k = 0$. The rank condition (17) is not a convex function constraint, which makes the reduced order H_∞ synthesis problem difficult to solve.

The rank condition can be, however, replaced by the trace equality constraint by using the following lemma:

Lemma 1 *Suppose $R \in \Re^{n\times n}$ and $S \in \Re^{n\times n}$ are symmetric and satisfy (16). Then $rank\,(RS - I) = 0$ if and only if $tr\,(RS) = n$.*

Therefore, there exists a constant matrix K^* such that $\|F_L(P_K(s), K^*)\|_\infty < \gamma$ if and only if

$$\min_{R,S} tr\,(RS) = n \quad \text{subject to (14)}\sim\text{(16)} \tag{18}$$

The cost function is still not a convex function of R and S. We employ the cone complementarity linearization algorithm [2] to solve this problem. By linearizing the cost function with respect to R and S, we have:

$$\min_{R_i, S_i} tr\,(S_{i-1}R_i + S_iR_{i-1}) \qquad \text{subject to (14)}\sim\text{(16)} \tag{19}$$

The procedure is initialized with R_0 and S_0 that satisfy LMIs (14)∼(16). With R_{i-1} and S_{i-1} fixed, R_i and S_i that minimize the trace function in (19) can be found by a convex optimization. By iteration over i, R_i and S_i will converge to one of the local minimums, or hopefully the global minimum, of the problem (18). Note that the cone complementarity algorithm is a local optimization algorithm and thus does not always guarantee to find the global minimum for (18). In usual cases, however, it shows excellent search performances, as is reported in [2] with extensive numerical examples.
The algorithm presented in this section does not include computation of the optimal K^* that achieves the closed-loop H_∞ norm (13) of γ. Once the optimal R and S that satisfy (18) are obtained, K^* can be computed by solving another convex optimization problem. See [1] for further detail.

3 Application to Steering Control of Lateral Motion of HDVs

3.1 Model Description

The linearized model of lateral motion of a single-unit heavy-duty vehicles (HDVs) is given in [3]:

$$\ddot{q} + A_{22}\dot{q} + A_{21}q = B_1\delta + B_2\dot{\epsilon}_d + B_3\ddot{\epsilon}_d \tag{20}$$

where

$$\begin{aligned} &A_{21} = M^{-1}K, A_{22} = M^{-1}D \\ &b_1 = M^{-1}F, b_2 = M^{-1}E_2, b_3 = M^{-1}E_2 \end{aligned} \tag{21}$$

and $q = [\ y_r \quad \epsilon_r \quad \epsilon_f\ ]^T$ is the generalized coordinate vector: y_r is the lateral displacement of the tractor's center of gravity, ϵ_r is the yaw angle of the tractor relative to the road, ϵ_f is the articulation angle between the tractor and semi-tractor. δ is the steering angle and it is the control input. y_r and ϵ_r are measured by using magnetometers attached at the front bumper of the tractor. $\dot{\epsilon}_d$ and $\ddot{\epsilon}_d$ are the yaw rate and yaw acceleration of the road frame, respectively. They are regarded as disturbances. See [3] for detailed descriptions of the inertial matrix M, the damping matrix D, the stiffness matrix K, coefficient matrices F, E_1 and E_2.

3.2 Optimization Problem

Control Objective: The control objective is to keep the lateral tracking error at the tractor center of gravity and the off-tracking error at the rear of the trailer to be small for lane-following maneuver. We need to tune the controller parameters (K_{p1},K_{i1},K_{p1},K_{i2}) (see below) such that the closed-loop system is stable and shows certain lane-following performances (i.e. the vehicle follows the center line of the road with a small error) even under model uncertainties or parameter perturbations.

Controller Structure: A two-inputs single-output PI controller of the following form is used to control the lateral motion of the vehicle:

$$\delta(s) = (K_{p1} + \frac{K_{i1}}{s + K_{T1}})(-y_r(s)) + (K_{p2} + \frac{K_{i2}}{s + K_{T2}})(-\epsilon_r(s)) \tag{22}$$

where K_{p1}, K_{i1}, K_{p1} and K_{i2} are tunable constant parameters. K_{T1} and K_{T2} are added to avoid a pure integrator. Here they are set to $K_{T1} = K_{T2} = 10^{-6}$.

Optimization Problem: By following the procedure discussed in Section 2.1, the extended plant $P_K(s)$ can be constructed such that $F_L(P_K(s), K)$ forms the closed-loop system with $K := \begin{bmatrix} K_{p1} & K_{i1} & K_{p1} & K_{i2} \end{bmatrix}$. Consider the following H_∞ optimization problem:

$$\text{find } K \in \Re^{1\times 4} \text{ such that } \left\| \begin{matrix} T^{cl}_{d\to y_r}(s)W_{p1}(s) \\ T^{cl}_{d\to \epsilon_r}(s)W_{p2}(s) \\ T^{cl}_{d\to \delta}(s)W_u(s) \end{matrix} \right\|_\infty \leq 1 \tag{23}$$

where $T^{cl}_{d\to y_r}(s)$, $T^{cl}_{d\to \epsilon_r}(s)$ and $T^{cl}_{d\to \delta}(s)$ denote the closed-loop transfer function from the disturbance d to the plant outputs y_s and ϵ_r and the control input δ, respectively. See Figure 3 for the closed-loop configuration. Our optimization objective can be stated as follows: We have desired "shapes" on closed-loop frequency responses from various reasons (specifically, those desired "shapes" are designed based on robustness and disturbance-rejection performance requirements for the closed-loop system). How should we set the controller parameters to make the actual closed-loop frequency responses look like the desired shapes?
This problem formulation is a standard H_∞ loop shaping problem formulation. $W_u(s)$, $W_{p1}(s)$ and $W_{p2}(s)$ define those desired "shapes" for each closed-loop frequency response. Here, $W_{p1}(s)$, $W_{p2}(s)$ and $W_u(s)$ are chosen as follows:

$$W_{p1}(s) = \frac{300(s + 5 \times 10^{-4})}{s + 0.1} \tag{24}$$

$$W_{p2}(s) = \frac{20(s + 6.667 \times 10^{-4})}{s + 0.1} \tag{25}$$

$$W_u(s) = \frac{9.8s^2 + 19.002s + 13.248}{s^2 + 24.090s + 13.380} \tag{26}$$

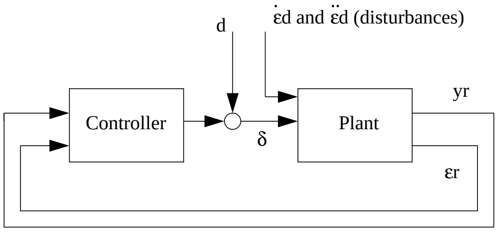

Figure 3: Closed-loop configuration for lateral motion control of the heavy-duty vehicle model.

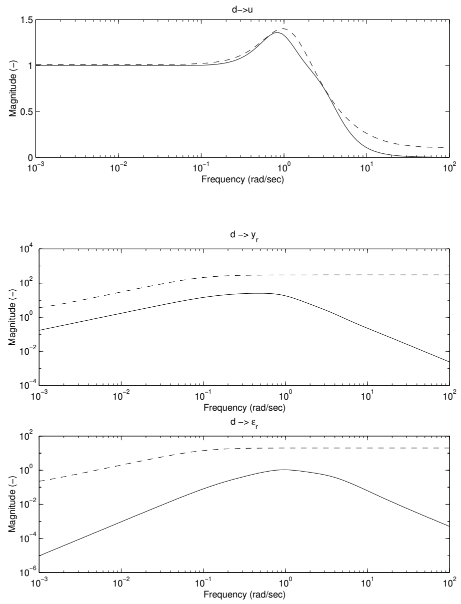

Figure 4: Frequency responses of the closed-loop system in the nominal condition, from $d(t)$ to $\delta(t)$ (upper), from $d(t)$ to $y_r(t)$ (middle) and from $d(t)$ to $\epsilon_r(t)$ (lower). The dashed lines are inverse frequency responses of the performance filters $W_{p1}(s)$, $W_{p2}(s)$ and $W_u(s)$, respectively.

3.3 Optimization Results

The optimal set of the controller parameters K can be computed by the LMI-based H_∞ optimization algorithm as presented in Section 2. After 15 iterations over R_i and S_i in (19), $tr(R_i S_i) = 12.000$ can be achieved (the weighted plant model P_k is 12th order; the original plant is 6th order, the controller adds two states and the filters are totally 4th order). The optimal gains $K_{p1} = 0.1997$, $K_{i1} = 0.0059$, $K_{p2} = 0.9546$ and $K_{i2} = -2.7723$ achieve the closed-loop H_∞ gain of 0.9992. Figure 4 (a)(b) show the closed loop frequency responses from $d(t)$ to $\delta(t)$ (upper), $y_r(t)$ (middle) and $\epsilon_r(t)$ (lower) in the nominal condition. The controller parameters are optimized such that each frequency response stays "under" that of the inverse of the corresponding performance filter.

3.4 Simulation Results

Time-domain simulations are conducted to show the closed-loop performance of the designed multi-inputs single-output (MISO) PI controller. The road curvature scenario for the simulations is designed based on the test track at Crows Landing Test Site [4]. Figure 5 shows the simulation results for the nominal condition and two perturbed conditions. y_{s1} is the lateral displacement at the tractor front axle and y_{s2} is the lateral displacement at the tractor rear axle. Smaller displacements means better lane following. The maneuver is accomplished with an overshoot from the lane centerline less than 0.5 m at the tractor's center of gravity (y_r) in the nominal condition, 2.5 m in the perturbed conditions.

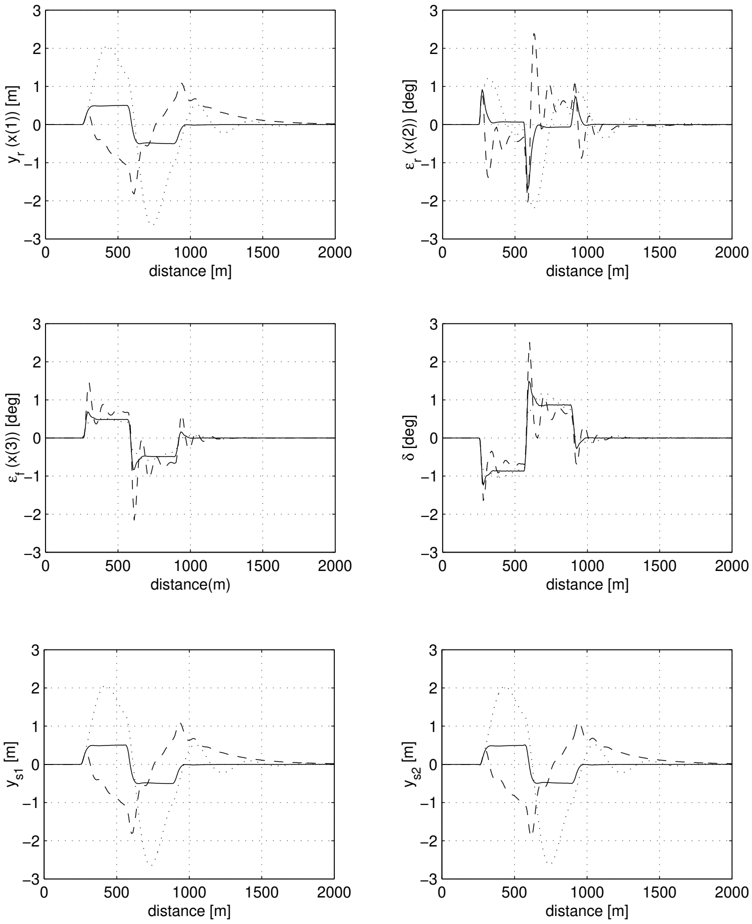

Figure 5: (a) Closed loop simulations: nominal condition (solid:v = 18 m/s, μ = 0.8, m_2 = 10670 kg) and perturbed conditions (dashed:v = 25 m/s, μ = 1.0, m_2 = 24000 kg) (dotted:v = 15 m/s, μ = 0.6, m_2 = 5000 kg)

4 Conclusion

In this paper, the H_∞ optimization algorithm for controller parameters of a fixed structure linear controller was presented. The proposed algorithm consists of two steps: 1) "Extraction" of the controller parameters from the closed-loop system as a full constant block by using LFTs and 2) the H_∞-optimal static output feedback controller synthesis. For the H_∞ static output feedback controller synthesis, the cone complementarity linearization algorithm is used to overcome the non-convexity that arises due to the constraint on the controller order. Although it is a local optimization algorithm, in most cases it shows excellent search performances.

As an application example, the proposed algorithm is applied to H_∞ optimization of four parameters of a MISO PI controller, which is employed for steering control of lateral motion of HDVs. Applied with the H_∞ loop-shaping techniques, the set of controller parameters are optimized such that the closed-loop system stays stable and achieves the desired disturbance rejection performances even under perturbed conditions. The results of time-domain simulations show the robust stability of the desired controller, despite its quite simple structure.

References

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