Homework Assignment #1

Due: Friday 9/3/99 (in my office, 267M)

Reading assignment: chapter 1 of *Course Reader*.

1. *Variational characterization of eigenvalues.*

   (a) Show that, for a square, symmetric matrix $A\ in \mathbf{R}^{n\times n}$, we have the variational characterization
   $$\lambda_{\max}(A) = \max\left\{x^T A x \mid \|x\|_2 = 1\right\}.$$

   (b) Show that
   $$\lambda_{\max}(A) = \max\left\{\mathbf{Tr}\, AX \mid \mathbf{Tr}\, X = 1, \;\; X \succeq 0\right\}.$$
   In other words, the maximum in the above problem is attained at a rank-one matrix.

   (c) Express similar results for the largest singular value of a rectangular matrix.

2. *Schur complements.* Consider a partioned symmetric matrix
   $$X = X^T = \left[\begin{array}{cc} A & B \\ B^T & C \end{array}\right]$$
   where A, C are square and symmetric as well. When A is invertible, we define $S := C - B^T A^{-1} B \succ 0$ and call it the *Schur complement* of A in X.

   (a) Show that $X \succ 0$ if and only if $A \succ 0$ and $S \succ 0$.
   (*hint:* consider the quadratic form associated with X and minimize it with respect to some variables)

   (b) Formulate a similar result in the case $X \succeq 0$ (but not necessarily definite).

3. *Dual maps.* For each of the following maps, define the input space, output space, and find the dual map.

   (a) $\Phi(X) = A^T X + XA$, where $A \in \mathbf{R}^{n\times n}$, and X is symmetric.

   (b) $\Phi(X) = X_{11}$, where X_{11} is the upper-left $r \times r$ block of $n \times n$ symmetric matrix X.

4. *Voronoi description of halfspace.* Let a and b be distinct points in $\mathbf{R}^n$. Show that the set of all points that are closer to a than b, *i.e.*, $\{x \| \|x - a\| \leq \|x - b\|\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

5. Which of the following sets are convex? If they are affine, cones, balls, ellipsoids, or polyhedra, say so.

   (a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n | \alpha \leq a^T x \leq \beta\}$.

   (b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n | \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

   (c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n | a_1^T x \leq b_1,\ a_2^T x \leq b_2\ \}$.

   (d) The set of points closer to a given point than a given set, *i.e.*,

   $$\{x | \|x - x_0\| \leq \|x - y\| \text{ for all } y \in S\ \}$$

   where $S \subseteq \mathbf{R}^n$.

   (e) The set of points closer to one set than another, *i.e.*, $\{x | \mathbf{dist}\,(x, S) \leq \mathbf{dist}\,(x, T)\ \}$, where $S, T \subseteq \mathbf{R}^n$, and $\mathbf{dist}\,(x, S) = \mathbf{inf}\{\|x - z\| \mid z \in S\ \}$.

   (f) The set $\{a \in \mathbf{R}^k\ | p(0) = 1,\ |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta\ \}$, where $p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1}$.

6. *Conic hulls of outer products.*

   (a) Consider the set of *outer products* or *dyads* in $\mathbf{R}^{n \times n}$: $\{xx^T | x \in \mathbf{R}^n\}$. Describe its conic hull in simple terms. What is its affine dimension?

   (b) Now consider the set of k-dimensional *outer products*, defined as $\{XX^T | X \in \mathbf{R}^{n \times k},\ \mathbf{Rank}(X) = k\}$. Describe its conic hull in simple terms.

7. *Expanded and restricted sets.* Let $S \subseteq \mathbf{R}^n$. For $a \geq 0$ we define S_a as $\{x | \mathrm{dist}(x, S) \leq a\}$. We refer to S_a as S expanded or extended by a. Show that is S is convex, then S_a is convex.

   For $a < 0$ we define $S_a = \{x | B(x, a) \subseteq S\}$. We refer to S_a as S shrunk or resticted by a; S_a consists of all points that are at least a distance a from $\mathbf{R}^n \backslash S$. Show that if S is convex and $a < 0$ then S_a is convex.

8. *More sets defined by distances to two points.*

   (a) Show that $\{x | \|x - a\| + \alpha \leq \|x - b\|\}$, *i.e.*, the set of points that are α closer to a than b, is convex if and only if $\alpha \geq 0$.

   (b) For what values of θ is the set $\{x | \|x - a\| \leq \theta \|x - b\|\}$ convex? For those values, give a simple description of the set.

   (c) Now we examine what happens if we use a different norm. Consider the two sets $\{x | \|x - a\|_\infty \leq \|x - b\|_\infty\}$ and $\{x | \|x - a\|_1 \leq \|x - b\|_1\}$. For each one either show it is convex, or give a counterexample.