Homework Assignment #2

Due: Wednesday September 8, 1999

Reading assignment: chapter 2 of *EE290N Course Reader.*

As in homework assignment 1, some of the problems are tough, so it's OK to skip a few parts if you're not getting any benefit from working on it any longer.

1. *Examples of convexity.* For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

   (a) $f(x) = e^x - 1$ on $\mathbf{R}_+$.

   (b) $f(x) = \max_{i=1,\ldots,k} \|A^{(i)}x - b^{(i)}\|$, where $A^{(i)} \in \mathbf{R}^{m\times n}$, $b^{(i)} \in \mathbf{R}^m$ and $x \in \mathbf{R}^n$.

   (c) $f(X) = \mathbf{Tr}\, X^{-1}$, on $\{X | X = X^T \succ 0\}$.

   (d) $f(x) = \frac{a^Tx+b}{c^Tx+d}$, on $\{x | c^Tx + d > 0\}$.

   (e) $\sum_{i=1}^n x_i \log x_i - (\sum_{i=1}^n x_i) \log(\sum_{i=1}^n x_i)$ on $x_i > 0$, $i = 1, \ldots, n$.
   Hint: show that

   $$\sum_{i=1}^n x_i \log \frac{x_i}{\sum_{i=1}^n x_i} = \begin{array}{ll} \text{minimum} & \sum_{i=1}^n x_i \log(x_i/y_i) \\ \text{subject to} & \sum_{i=1}^n y_i = 1, \quad y_i \geq 0, \quad i = 1, \ldots, n. \end{array}$$

   (f) $f(x) = \sum_{i=1}^r |x|_{[i]}$ on $\mathbf{R}^n$, where $|x|$ denotes the vector with $|x|_i = |x_i|$ (*i.e.*, $|x|$ is the absolute value of x, componentwise), and $|x|_{[i]}$ is the ith largest component of $|x|$.

2. *Convex-over-concave functions.* Suppose $f(x) = g(x)/h(x)$, with

   $$\mathbf{dom}\, f = \{x \in \mathbf{dom}\, g \cap \mathbf{dom}\, h | h(x) > 0\},$$

   where $g : \mathbf{R}^n \to \mathbf{R}$ is convex and nonnegative, and $h : \mathbf{R}^n \to \mathbf{R}$ is concave. Show that f is quasiconvex.

3. *Some functions on the probability simplex.* Let x be a real-valued random variable which takes values in $\{a_1, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$, with $\mathbf{Prob}(x = a_i) = p_i$, $i = 1, \ldots, n$. For each of the following functions of p on the probability simplex $\{p \in \mathbf{R}^n \mid p_i \geq 0, \sum_i^n p_i = 1\}$, is it convex, concave, quasiconvex, quasiconcave?

   (a) $\mathbf{E}x$

   (b) $\mathbf{Prob}(x \geq \alpha)$

   (c) $\mathbf{Prob}(\alpha \leq x \leq \beta)$

(d) $\sum_i p_i \log p_i$ (the entropy of the distribution)

(e) $\mathbf{var} x$

(f) Quartile(x)

(g) the size of the smallest set $\mathcal{A} \subseteq \{a_1, \ldots, a_n\}$ with probability $\geq 90\%$ (By size we mean the number of elements in $\mathcal{A}$.)

4. *Matrix convexity.* Show that the function $f(X) = X^{-1}$ is matrix-convex on the positive definite cone. (Matrix-convexity is defined on p.65 of the course reader.)