Homework Assignment #3

Due: Wednesday September 22, 1999

Reading assignment: parts 3.1, 3.2 and 3.3 in chapter 3 of *Course Reader*.

1. *More convex functions.*
   (a) Show that the function
   $$f(x,t) = -\log(t^2 - x^T x)$$
   is convex on the set $\{(t,x) \mid t > \|x\|\}$.
   (b) Show the function $f(X,Y) = \mathbf{Tr}\, Y^T X^{-1} Y$ defined on $\{(X,Y) \mid Y \in \mathbf{R}^{n\times k},\ \ X \succ 0\}$, is convex.
2. *Examples of conjugate functions.*
   (a) Show that the conjugate of the indicator function of a convex set C,
   $$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C \\ 1 & \text{else} \end{cases}$$
   is its support function $s_C(x) = \mathbf{sup}_{y\in C}\, y^T x$.
   (b) Conversely, show that if C is closed, then the conjugate of its support function is its indicator function.
   Hint: Use the strong separation theorem between a closed convex set and a point, mentioned in the notes (page 23).
   (c) Find the conjugate function of the largest eigenvalue function $\lambda_{\mathbf{max}}(X)$, defined on $\{X \mid X = X^T \in \mathbf{R}^{n\times n}\}$.
3. *Subgradients and quasigradients.* For each of the following functions, give the expression of subgradients, or quasigradients (when the function is quasiconvex only).
   (a) The quadratic function
   $$f(x) = x^T A x + 2x^T b + c,$$
   where $A = A^T \succ 0$.
   (b) The pointwise supremum of a finite family of convex functions, $\max(f_1, \ldots, f_n)$.

(c) The largest generalized eigenvalue, $f(X, Y) = \lambda_{\max}(X, Y)$ on the set $\{X = X^T,\ Y = Y^T \in \mathbf{R}^{n\times n},\ \mid Y \succ 0\}$.

Hint: show (using the results of homework 2, problem 1) that

$$\lambda_{\max}(X, Y) = \mathop{\mathbf{sup}}_{Z} \left\{ \frac{\mathbf{Tr}\, XZ}{\mathbf{Tr}\, YZ} \mid Z \geq 0, \quad Z \neq 0 \right\},$$

and that the supremum is achieved at some matrix $Z \succeq 0$, $Z \neq 0$.

4. *Approximating a matrix in infinity norm.* The infinity (induced) norm of a matrix $A \in \mathbf{R}^{m\times n}$, denoted $\|A\|_{\infty,i}$, is defined as

$$\|A\|_{\infty,i} = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|.$$

The infinity norm gives the maximum ratio of the infinity norm of Ax to the infinity norm of x:

$$\|A\|_{\infty,i} = \mathop{\mathbf{sup}}_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}.$$

This norm is sometimes called the max-row-sum norm, for obvious reasons.

Consider the problem of approximating a matrix, in the max-row-sum norm, by a linear combination of other matrices. That is, we are given $k+1$ matrices $A_0, \dots, A_k \in \mathbf{R}^{m\times n}$, and need to find $x \in \mathbf{R}^k$ that minimizes

$$\|A_0 + x_1 A_1 + \cdots + x_k A_k\|_{\infty,i}.$$

Express this problem as a linear program, with a number of constraints and variables that grows only polynomially with problem size (*i.e.*, $m + n$). Explain the significance of any extra variables in your LP. Carefully explain why your LP formulation solves this problem, *e.g.*, what is the relation between the feasible set for your LP and this problem?