EE290N L. El Ghaoui 9/22/99

Homework Assignment #4

Due: Friday October 1, 1999 (note the extra time)

Reading assignment: rest of chapter 3 of *Course Reader*.

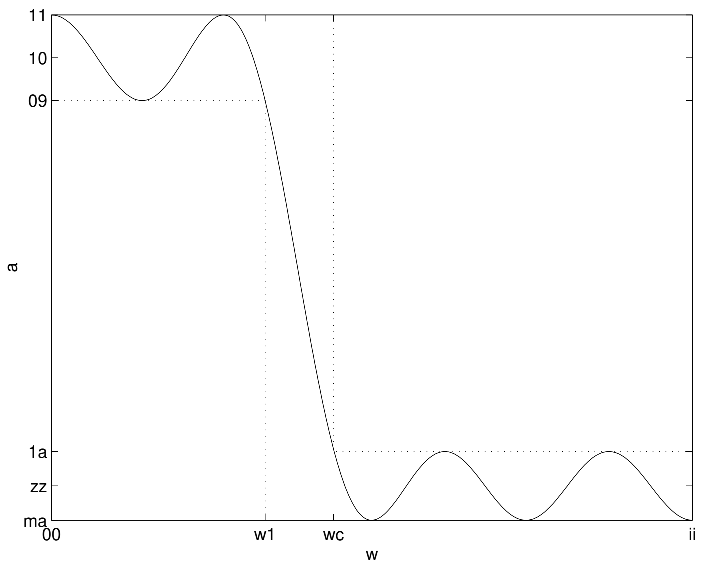
1. *FIR filter design.* Consider the (symmetric, linear phase) FIR filter described by

$$H(\omega) = a_0 + \sum_{k=1}^{N} a_k \cos k\omega.$$

The design variables are the real coefficients $a = (a_0, \ldots, a_N) \in \mathbf{R}^{N+1}$. In this problem we will explore the design of a low-pass filter, with specifications:

- For $0 \leq \omega \leq \pi/3$, $0.89 \leq H(\omega) \leq 1.12$, *i.e.*, the filter has about ± 1dB ripple in the 'passband' $[0, \pi/3]$.
- For $\omega_c \leq \omega \leq \pi$, $|H(\omega)| \leq \alpha$. In other words, the filter achieves an attenuation given by α in the 'stopband' $[\omega_c, \pi]$. ω_c is called the 'cutoff frequency'.

These specifications are depicted graphically in the figure below.



(a) Suppose we fix ω_c and N, and wish to maximize the stop-band attenuation, *i.e.*, minimize α such that the specifications above can be met. Explain how to pose this problem as a convex programming problem.

(b) Suppose we fix N and α, and want to minimize ω_c, *i.e.*, we set the stopband attenuation and filter length, and wish to minimize the 'transition' band (between $\pi/3$ and ω_c). Explain how to pose this problem as a quasiconvex programming problem.

(c) Now suppose we fix ω_c and α, and wish to find the smallest N that can meet the specifications, *i.e.*, we seek the shortest length FIR filter that can meet the specifications. Can this problem be posed as a convex or quasiconvex problem? If so, explain how. If you think it cannot be, briefly and informally explain why.

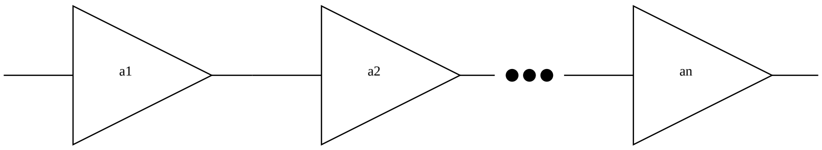
(d) Plot the optimal tradeoff curve of attenuation (α) versus cutoff frequency (ω_c) for $N = 7$. Is the set of achievable specifications convex? Briefly explain any interesting features, *e.g.*, flat portions, of the optimal tradeoff curve.

For this subproblem, you may sample the constraints in frequency, which means the following. Choose $K \gg N$ (perhaps $K \approx 10N$), and set $\omega_k = k\pi/K$, $k = 0, \ldots, K$. Then replace the specifications with

- For k with $0 \le \omega_k \le \pi/3$, $0.89 \le H(\omega_k) \le 1.12$.
- For k with $\omega_c \le \omega_k \le \pi$, $|H(\omega_k)| \le \alpha$.

With this approximation, the problem in part (a) becomes an LP, which allows you to solve part (d) numerically using matlab's `linprog`.

2. *Optimal amplifier gains.* We consider a system of n amplifiers connected (for simplicity) in a chain, as shown below. The variables that we will optimize over are the gains $a_1, \ldots, a_n > 0$ of the amplifiers. The first specification is that the overall gain of the system, *i.e.*, the product $a_1 \cdots a_n$, is equal to A, which is given.



We are concerned about two effects: noise generated by the amplifiers, and amplifier overload. These effects are modeled as follows.

We first describe how the noise depends on the amplifier gains. Let N_i denote the noise level (RMS, or root-mean-square) at the output of the ith amplifier. These are given recursively as

$$N_0 = 0, \quad N_i = a_i \left(N_{i-1}^2 + \alpha_i^2\right)^{1/2}, \quad i = 1, \ldots, n$$

where $\alpha_i > 0$ (which is given) is the ('input-referred') RMS noise level of the ith amplifier. The *output noise level* N_{out} of the system is given by $N_{\text{out}} = N_n$, *i.e.*, the noise level of the last amplifier. Evidently N_{out} depends on the gains $a_1, \ldots, a_n$.

Now we describe the amplifier overload limits. S_i will denote the (RMS) signal level at the output of the ith amplifier. These signal levels are related by

$$S_0 = S_{\text{in}}, \quad S_i = a_i S_{i-1}, \qquad i = 1, \ldots, n,$$

where $S_{\text{in}} > 0$ is the *input signal level.* Each amplifier has a maximum allowable output level $M_i > 0$ (which is given). (If this level is exceeded the amplifier will distort the signal.) Thus we have the constraints $S_i \leq M_i$, for $i = 1, \ldots, n$. (A more accurate model would include the noise as well as the signal in the overload condition, but since the signal levels are usually much larger than the noise levels, we can reasonably neglect it here.)

The *maximum output signal level* S_{max} is defined as the maximum value of S_n, over all input signal levels S_{in} that respect the the overload constraints $S_i \leq M_i$. Of course $S_{\text{max}} \leq M_n$, but it can be smaller, depending on the gains $a_1, \ldots, a_n$.

The *dynamic range* D of the system is defined as $D = S_{\text{max}}/N_{\text{out}}$. Evidently it is a (rather complicated) function of the amplifier gains $a_1, \ldots, a_n$.

The goal is to choose the gains a_i to maximize the dynamic range D, subject to the constraint $\prod_i a_i = A$, and upper bounds on the amplifier gains, *i.e.*, $a_i \leq A_i$ (which are given).

Explain how to solve this problem as a convex (or quasiconvex) optimization problem. If you introduce new variables, or transform the variables, explain. Clearly give the objective and inequality constraint functions, explaining why they are convex if it is not obvious. If your problem involves equality constraints, give them explicitly.

3. *A quadratic problem.* Consider the minimization problem in variable $x \in \mathbf{R}^n$

$$\text{minimize} \begin{bmatrix} x \\ 1 \end{bmatrix}^T F^0 \begin{bmatrix} x \\ 1 \end{bmatrix} \text{ subject to } \begin{bmatrix} x \\ 1 \end{bmatrix}^T F^k \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0, \quad k = 1, \ldots, m,$$

where the $(n+1) \times (n+1)$ symmetric $F^0, \ldots, F^m$ are positive *componentwise,* meaning $F^k_{ij} \geq 0$ for every $1 \leq i, j \leq n$ and $1 \leq k \leq m$.

(a) Is the above problem a convex problem in general? (Prove it or find a counterexample).

(b) Show how, by a change of variables, we can make it a convex programming problem of a type encountered recently in the course.