Homework Assignment #4 Solutions

1. *FIR filter design.*

   (a) Given ω_c and N, let $(a, \alpha) \in \mathbf{R}^{N+1} \times \mathbf{R}_+$ be the design variable. The problem

   $$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & 0.89 \leq H(\omega) \leq 1.12, \ \ \forall\, \omega \\ & -\alpha \leq H(\omega) \leq \alpha, \ \ \forall\, \omega \in [\omega_c, \pi] \end{array}$$

   is in fact equivalent to the problem $(\mathbf{P}_1)$:

   $$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & (a, \alpha) \in C \end{array}$$

   where $C = \left(\bigcap_{\omega \in [0, \pi/3]} C_{1,\omega}\right) \bigcap \left(\bigcap_{\omega \in [\omega_c, \pi]} C_{2,\omega}\right)$ with

   $$\begin{array}{rcl} C_{1,\omega} & = & \{\ (a, \alpha) \mid 0.89 \leq a_0 + \sum_{k=1}^{N} a_k \cos k\omega \leq 1.12\ \} \\ & = & \{\ (a, \alpha) \mid 0.89 \leq [1\ \ \cos\omega\ \ \cdots\ \ \cos N\omega] a \leq 1.12\ \}, \end{array}$$

   and

   $$\begin{array}{rcl} C_{2,\omega} & = & \{\ (a, \alpha) \mid -\alpha \leq a_0 + \sum_{k=1}^{N} a_k \cos k\omega \leq \alpha\ \} \\ & = & \{\ (a, \alpha) \mid -\alpha \leq [1\ \ \cos\omega\ \ \cdots\ \ \cos N\omega] a \leq \alpha\ \}. \end{array}$$

   Clearly $C_{1,\omega}$ and $C_{2,\omega}$ are convex sets (in fact, slabs) for each ω. Thus $(\mathbf{P}_1)$ is a convex programming problem because it has a convex (linear) objective and convex constraints.

   Note that they are not necessarily easy to solve, because the convex constraint set is described via an infinite number of linear inequalities.

   (b) Given α and N, let $(a, \omega_c) \in \mathbf{R}^{N+1} \times [0, \pi]$ be our design variable and we have the problem

   $$\begin{array}{ll} \text{minimize} & \omega_c \\ \text{subject to} & 0.89 \leq H(\omega) \leq 1.12, \ \ \omega \in [0, \pi/3] \\ & -\alpha \leq H(\omega) \leq \alpha, \ \ \omega \in [\omega_c, \pi]. \end{array}$$

   The above problem is equivalent to the problem $(\mathbf{P}_2)$:

   $$\begin{array}{ll} \text{minimize} & f(a) \\ \text{subject to} & (a, \omega_c) \in C, \end{array}$$

   where $f : \mathbf{R}^{N+1} \to [0, \pi]$

   $$f(a) = \omega_c = \inf \{\ \omega \mid -\alpha \leq H(\omega') \leq \alpha,\ 0 \leq \omega \leq \omega' \leq \pi\ \}$$

To show $(\mathbf{P}_2)$ is a quasiconvex programming problem, we only need to show that f is quasiconvex. Let S_Ω denote the Ω-sublevel set of f. We have

$$S_\Omega = \{ a \mid f(a) \leq \Omega \}$$

Now, we take two vectors $a_1, a_2 \in S_\Omega$ and note that, for every convex combination $\theta a_1 + (1-\theta)a_2$ the filter $\theta H_1 + (1-\theta)H_2$ is also in S_Ω (where H_1 is a filter such that $-\alpha \leq H_1(\omega) \leq \alpha$ for $w \geq \Omega$, that is, H_1 is a filter defined by the coefficients a_1, and respectively for H_2.) Hence the sublevel sets of f are convex, f is quasiconvex and $(\mathbf{P}_2)$ is a quasiconvex programming problem.

The result is in fact an application of the nested set characterization of quasiconvex functions, see the notes p. 41.

(c) Now given α and ω_c, the design variable becomes (a, N). We would like to solve

$$\begin{array}{ll} \text{minimize} & N \\ \text{subject to} & 0.89 \leq H(\omega) \leq 1.12, \quad \omega \in [0, \pi/3] \\ & -\alpha \leq H(\omega) \leq \alpha, \quad \omega \in [\omega_c, \pi]. \end{array}$$

We show that N is a quasiconvex function of the filter coefficients a and therefore minimizing N with fixed ω_c and α is a quasiconvex optimization problem. Note that the constraints are already known to be convex (they are an intersection of an infinite number of halfspaces).

Without loss of generality, let $a \in \mathbf{R}^\infty$. In this case, $N(a)$, the degree of the FIR filter with filter coefficients $a \in \mathbf{R}^\infty$ can be written as

$$N(a) = \inf \{ \, i \,| a_j = 0 \text{ for } j \geq i \, \}.$$

Clearly, $N : \mathbf{R}^\infty \to \mathbf{Z}_+$ is a quasiconvex function because for any $N_0 \in \mathbf{Z}_+$, the N_0-sublevel set S_{N_0} is convex. Note that

$$\begin{aligned} S_{N_0} &= \{ a \mid N(a) \leq N_0 \} \\ &= \{ a \mid \inf \{ \, i \,| a_j = 0 \text{ for } j \geq i \, \} \leq N_0 \} \\ &= \{ a \mid a_j = 0 \text{ for } j \geq N_0 \} \end{aligned}$$

which is nothing but an intersection of halfspaces in $\mathbf{R}^\infty$. Hence S_{N_0} is a convex set and therefore N is a quasiconvex function.

(d) To obtain the optimal tradeoff curve of α versus ω_c, we minimize α such that the filter specifications are met for $N = 7$ and various $\omega_c \in [0, \pi]$. As shown in part (a), the problem of minimizing α for fixed ω_c and N is a convex optimization problem:

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & 0.89 \leq H(\omega) \leq 1.12, \quad \omega \in [0, \pi/3] \\ & -\alpha \leq H(\omega) \leq \alpha, \quad \omega \in [\omega_c, \pi] \end{array}$$

or

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & 0.89 \leq [1 \;\; \cos\omega \;\; \cdots \;\; \cos N\omega] a \leq 1.12, \quad \omega \in [0, \pi/3] \\ & -\alpha \leq [1 \;\; \cos\omega \;\; \cdots \;\; \cos N\omega] a \leq \alpha, \quad \omega \in [\omega_c, \pi] \end{array}$$

and finally

$$
\begin{array}{ll}
\text{minimize} & [0\ 1]\begin{bmatrix} a \\ \alpha \end{bmatrix} \\
\text{subject to} & [1\ \ \cos\omega\ \cdots\ \cos N\omega\ \ 0]\begin{bmatrix} a \\ \alpha \end{bmatrix} \ \leq\ 1.12, \quad \omega\in[0,\pi/3] \\
& -[1\ \ \cos\omega\ \cdots\ \cos N\omega\ \ 0]\begin{bmatrix} a \\ \alpha \end{bmatrix} \ \leq\ -0.89, \quad \omega\in[0,\pi/3] \\
& [1\ \ \cos\omega\ \cdots\ \cos N\omega\ \ -1]\begin{bmatrix} a \\ \alpha \end{bmatrix} \ \leq\ 0, \quad \omega\in[\omega_c,\pi] \\
& -[1\ \ \cos\omega\ \cdots\ \cos N\omega\ \ 1]\begin{bmatrix} a \\ \alpha \end{bmatrix} \ \leq\ 0, \quad \omega\in[\omega_c,\pi].
\end{array}
$$

Clearly, this is a linear program in the variables $[a^T\ \alpha]^T$ but with an *infinite* number of inequality constraints. In order to be able to (approximately) solve this infinite-dimensional LP (for example using `lp`), we can discretize the intervals $[0,\pi/3]$ and $[\omega_c,\pi]$ to get an (approximate) LP with a finite number of constraints, *i.e.*,

$$
\begin{array}{ll}
\text{minimize} & [0\ 1]\begin{bmatrix} a \\ \alpha \end{bmatrix} \\
\text{subject to} & [1\ \ \cos\omega\ \cdots\ \cos N\omega\ \ 0]\begin{bmatrix} a \\ \alpha \end{bmatrix} \ \leq\ 1.12, \\
& \qquad \text{for}\ \ \omega = 0, \frac{\pi}{3M}, \ldots, \frac{(M-1)\pi}{3M}, \frac{\pi}{3} \\
& -[1\ \ \cos\omega\ \cdots\ \cos N\omega\ \ 0]\begin{bmatrix} a \\ \alpha \end{bmatrix} \ \leq\ -0.89, \\
& \qquad \text{for}\ \ \omega = 0, \frac{\pi}{3M}, \ldots, \frac{(M-1)\pi}{3M}, \frac{\pi}{3} \\
& [1\ \ \cos\omega\ \cdots\ \cos N\omega\ \ -1]\begin{bmatrix} a \\ \alpha \end{bmatrix} \ \leq\ 0, \\
& \qquad \text{for}\ \ \omega = \omega_c, \omega_c + \frac{(\pi-\omega_c)}{M}, \ldots, \pi \\
& -[1\ \ \cos\omega\ \cdots\ \cos N\omega\ \ 1]\begin{bmatrix} a \\ \alpha \end{bmatrix} \ \leq\ 0, \\
& \qquad \text{for}\ \ \omega = \omega_c, \omega_c + \frac{(\pi-\omega_c)}{M}, \ldots, \pi,
\end{array}
$$

where M is a large enough number (say $M \approx 100$).

In order to use Matlab's `lp` command, we need to put the LP in the following standard form

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \preceq b
\end{array}
$$

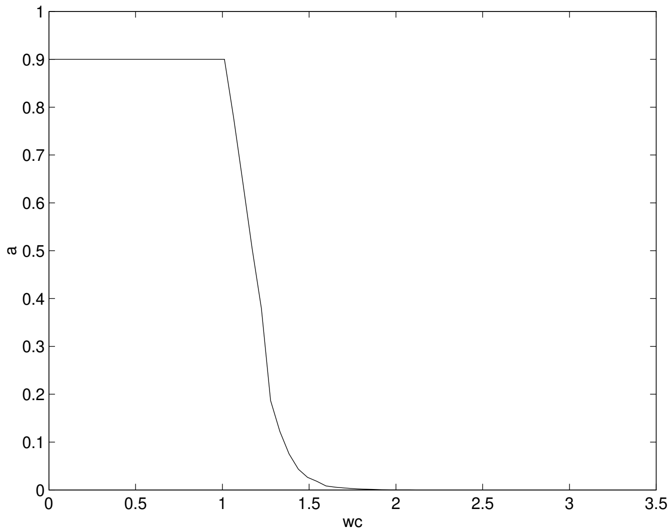
This can be done using the following commands in Matlab (given `wc`):

```
N=7;
M=200;

c=[zeros(N+1,1);1];
b=[1.12*ones(M+1,1);-0.89*ones(M+1,1);zeros(2*(M+1),1)];

w1=linspace(0,pi/3,M+1)';
w2=linspace(wc,pi,M+1)';
nn=linspace(1,N,N);

W1=w1*nn;
W2=w2*nn;

C1=cos(W1);
C2=cos(W2);

A=[ones(M+1,1),C1,zeros(M+1,1);-ones(M+1,1),-C1,-zeros(M+1,1); ...
   ones(M+1,1),C2,-ones(M+1,1);-ones(M+1,1),-C2,-ones(M+1,1)];
```

Now `x=lp(c,A,b)` gives the optimum values for $a \in \mathbf{R}^{N+1}$ and $\alpha \in \mathbf{R}$. In Matlab:

```
a=x(1:N+1,1);
alpha=x(N+2,1);
```

For different values of ω_c we solve the corresponding LP in Matlab and plot the optimum α vs. ω. The tradeoff curve looks like



From the figure we can clearly see that the region of achievable objectives is *not* convex. This provides an example showing that if the objectives in a multicriterion optimization problem are only quasiconvex, and not convex, then the region of achievable objectives need not be convex. Note also that the optimal α is equal to 0.89 for $\omega_c \leq \pi/3$ in order to satisfy the constraint

$$0.89 \leq H(\omega) \leq 1.12, \quad \omega \in [0, \pi/3].$$

2. *Optimal amplifier gains.* We have the constraints

$$a_1 \cdots a_n = A, \quad a_i \leq A_i, \quad i = 1, \ldots, n.$$

To get an expression for the output noise level, we first write the recursion as

$$N_0^2 = 0, \quad N_i^2 = a_i^2 \left(N_{i-1}^2 + \alpha_i^2 \right), \quad i = 1, \ldots, n.$$

Then we can express the square of the output noise level (*i.e.*, the output noise power) as

$$N_{\text{out}}^2 = \sum_{i=1}^{n} \prod_{j=i}^{n} a_j^2 \alpha_i^2.$$

The amplifier overload constraints can be expressed as

$$S_i = S_{\text{in}} \prod_{j=1}^{i} a_j \leq M_i, \qquad i = 1, \ldots, n.$$

Hence the maximum possible input signal level is

$$S_{\text{in,max}} = \min_i \frac{M_i}{\prod_{j=1}^{i} a_j}$$

Since $S_n = AS_{\text{in}}$, we can express the maximum output signal level as

$$S_{\max} = AS_{\text{in,max}} = \min_i \; M_i \prod_{j=i+1}^{n} a_j$$

(we interpret $\prod_{n+1}^{n}$ as 1).

Therefore the dynamic range is given by

$$D = \frac{\min_i \; M_i \prod_{j=i+1}^{n} a_j}{\left(\sum_{i=1}^{n} \prod_{j=i}^{n} a_j^2 \alpha_i^2 \right)^{1/2}}.$$

Wow! That is not exactly what you'd call a simple function of a_i.

After taking a deep breath and relaxing, you will realize this is in fact a geometric program. Let $a_i = \exp x_i$. Then the gain limits and overall gain constraint become

$$\sum_i x_i = \log A, \quad x_i \leq \log A_i, \tag{1}$$

i.e., one linear equality constraint and n linear inequalities. Now note that $D \geq t$ if and only if

$$M_k \prod_{j=k+1}^{n} a_j \geq t \left(\sum_{i=1}^{n} \prod_{j=i}^{n} a_j^2 \alpha_i^2 \right)^{1/2}, \quad k = 1, \ldots, n,$$

which we can rewrite as

$$\frac{t^2 \sum_{i=1}^{n} \prod_{j=i}^{n} a_j^2 \alpha_i^2}{M_k^2 \prod_{p=k+1}^{n} a_p^2} \leq 1, \quad k = 1, \ldots, n.$$

The left hand sides are complicated, but posynomial, functions of x_i and t. Hence with $t = \exp u$ these inequalities become convex in x_i and t.

Thus, the problem becomes: maximize t subject to the constraints (1), and the convex constraints which result from the posynomial constraints above. The variables are t and x_i; this is convex optimization problem since t is a linear, hence concave, function of the variables.

3. *A quadratic problem.* Unfortunately, the problem does not make sense! For every $x_i > 0$, and matrix $F = F^T$ with every $F_{ij} \geq 0$, the quadratic form

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T F \begin{bmatrix} x \\ 1 \end{bmatrix}$$

is non-negative. I apologize for the time you might have spent on this meaningless problem ...

Let me give you an idea of what I was after. I thought we could find a meaningful quadratic problem that is not convex, but becomes convex after the change of variables $y_i = \log x_i$. In other words, I was looking for quadratic problems that are geometric programs. It seems that there are no such problems (except trivial cases). Any ideas?