Homework Assignment #5

Due: Friday October 8, 1999 (before 2:00 pm please)

Reading assignment: chapter 4 of *Course Reader*.

1. *Dual of SOCP.* Consider the SOCP with inequality constraints only

$$\begin{array}{ll} \text{minimize} & e^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, N. \end{array}$$

   Find the dual of this problem by following these steps:

   - Introduce new variables y, t and equality constraints $y_i = A_i x + b_i$ and $t_i = c_i^T x + d_i$. You might find useful to write those constraints in a more compact way $y = Ax + b$, $t = C^T x + d$, where A, b, C, d are appropriate matrices and vectors.
   - Find the Lagrange dual function of the resulting problem.
   - Form the Lagrange dual problem, making any implicit equality constraints explicit.

2. *Portfolio optimization under incomplete covariance information.* We consider the problem of investing a sum S in a number of n assets over a given investment period. Over the period, the price of asset i changes by an amount given by p_i, with p modeled as a random vector, with mean and and covariance

$$\overline{p} = \mathbf{E}p, \quad \Sigma = \mathbf{E}(p - \overline{p})(p - \overline{p})^T.$$

   Therefore, the change in value of the portfolio rate of return of the portfolio is a random variable with mean $\overline{p}^T x$ and standard deviation $\sigma = \left(x^T \Sigma x\right)^{1/2}$.

   Let x be a vector that represents the proportion of the sum S invested in asset i. We define the set of feasible investment strategies as a polytope $\mathcal{X} = \{x \mid Ax \preceq b\}$, where A is a given matrix and b a given vector. $\mathcal{X}$ might be, for example, the set

$$\mathcal{X} = \left\{x \mid x_1 + \ldots + x_n = S, \;\; x_i \ge 0, \;\; \overline{p}^T x \ge r\right\}, \tag{1}$$

   where r is a given lower bound on the mean return.

   Markowitz (1950) has suggested the following strategy for choosing x. Let us minimize the risk of a large loss, as measured by the variance. The problem is formulated as

$$\text{minimize } x^T \Sigma x \text{ subject to } x \in \mathcal{X}$$

For fixed $\Sigma \succeq 0$, this is a convex, quadratic program.

In this problem we assume only partial information is available about the covariance matrix Σ, *i.e.*, $\Sigma \in \mathcal{E}$, where $\mathcal{E}$ is the set of possible values of the covariance. For a given investment strategy $x \in \mathcal{X}$, the maximum level of risk for our portfolio, over all possible covariances, is given by

$$\sigma_{\mathrm{wc}}(x) := \max \left\{ \left(x^T \Sigma x\right)^{1/2} \mid \Sigma \in \mathcal{E} \right\}.$$

(The subscript stands for "worst-case".)

We'll take a simple case as an example. We suppose that Σ_{ii}, which is related to the price volatility of asset i, are known. For the off-diagonal entries of Σ, all we know is the sign (or, in some cases, nothing at all). For example, we might be given that $\Sigma_{12} \geq 0$, $\Sigma_{23} \leq 0$, etc. This means that we do not know the correlation between p_1 and p_2, but we do know that they are nonnegatively correlated (*i.e.*, the prices of assets 1 and 2 tend to rise or fall together). And, of course, we know that $\Sigma \succeq 0$, *i.e.*, the covariance matrix, whatever it is, is positive semidefinite.

(a) Assume first x is fixed. Show how to compute $\sigma_{\mathrm{wc}}(x)$, using semidefinite programming, for the partial information on Σ described in the paragraph above.

You might find useful to denote by I_0, I_+ and I_- the sets of index pairs (i,j) for which Σ_{ij} is known, only known to be nonnegative, and nonpositive, respectively. Also denote by Σ^0 the matrix that contains the known elements $\Sigma^0_{ij} = \Sigma_{ij}$ if $(i,j) \in I_0$, and zero otherwise.

(b) Compute $\sigma_{\mathrm{wc}}(x)$ for the specific case

$$x = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.05 \\ 0.1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.2 & + & + & \pm \\ + & 0.1 & - & - \\ + & - & 0.3 & + \\ \pm & - & + & 0.1 \end{bmatrix},$$

where a "+" entry means that the element is nonnegative, a "−" means the entry is nonpositive, and "±" means we don't know anything about the entry.

In addition to σ_{wc}, give the covariance matrix Σ_{wc} associated with the maximum risk. Be sure to verify that $\Sigma_{\mathrm{wc}} \in \mathcal{E}$.

(c) What is the dual of the problem? Express it as an SDP in matrices P^0, P^+, P^- that are zero whenever $(i,j) \notin I_0$, I_+ and I_-, respectively.

(d) We finally turn to the problem of choosing x, so as to minimize the worst-case variance of the portfolio. The problem reads

$$\text{minimize} \max_{\Sigma \in \mathcal{E}} x^T \Sigma x \text{ subject to } x \in \mathcal{X}.$$

Using duality, show how formulate the problem as an SDP.

(e) Solve the problem using the numerical example above, and with $\mathcal{X}$ given by (1), with $S = 1000$, $r = 100$, and $\overline{p} = \begin{bmatrix} 2 & 3 & 1 & 2 \end{bmatrix}^T$.