Homework Assignment #51

The total number of points is 30. You can use whatever notes you have.

1. *Dual of general LP.* (6 pts) Consider the linear program with equality and inequality constraints,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h, Ax = b. \end{array}$$

   (a) Find the Lagrangian for this problem.

   (b) Find the dual function g for this problem. Be sure to indicate when the dual function is finite, and when it is $-\infty$.

   (c) Give the Lagrange dual of the problem, after making the implicit equality constraints explicit.

2. *A quadratic optimization problem.* (12 pts) For a given $n \times n$ matrix $W = W^T \succ 0$, we consider the quadratic optimization problem

$$\text{maximize } x^T W x \text{ subject to } x_i^2 \leq 1, \;\; 1 \leq i \leq m. \tag{1}$$

   We address the problem of computing an upper bound on its optimal value via SDP.

   (a) Find a Lagrange function $L(x, \lambda)$ for the problem, such that

$$p^* = \sup_x \inf_{\lambda \succeq 0} L(x, \lambda).$$

   (b) Show that the optimal value of problem (1) satisfies

$$p^* \leq d^* := \inf_{\lambda \succeq 0} \sup_x L(x, \lambda).$$

   (c) Show that the upper bound d^* can be computed by solving the SDP

$$\text{minimize } \mathbf{Tr}\, S \text{ subject to } S \text{ diagonal}, \; W \preceq S. \tag{2}$$

   (d) Using $X = xx^T$, show that problem (1) can be written as

$$\max \mathbf{Tr}\, WX \text{ subject to } X_{ii} = 1, \; 1 \leq i \leq n, \;\; X \succeq 0, \;\; \mathbf{Rank} X = 1.$$

   (e) Explain how to relax the above problem to obtain an SDP.

   (f) Show that the SDP obtained previously is dual to problem (2).

3. *LP with implementation constraints.* (8 pts) We consider the linear programming problem

$$\text{minimize } c^T x \text{ subject to } a_i^T x \leq b_i, \quad i = 1, \ldots, m.$$

We assume that the optimal variable x^{opt} will be implemented with some relative error. (Think of x as containing transistor widths in a VLSI design problem, that is implemented to some degree of relative accuracy.) The implemented variable will be of the form $x^{\text{impl}} = x^{\text{opt}} + \delta$, where $\|\delta\|_2 \leq 0.01\|x\|_2$.

(a) We observe the following fact. The optimal solution is obtained with no difficulty, but when we implement it, the performance of the solution (measured via the objective function $c^T x^{\text{impl}}$) jumps at some unacceptable value. We even observe that x^{impl} is not feasible. Briefly explain this fact, preferably using a picture.

(b) We consider the problem

$$\min \max_{\delta \in \mathcal{B}(x)} c^T(x + \delta) \text{ subject to: } a_i^T(x + \delta) \leq b_i \text{ for every } \delta \in \mathcal{B}(x), \quad 1 \leq i \leq m,$$

where $\mathcal{B}(x)$ denotes the "error set" $\{\delta \mid \|\delta\|_2 \leq 0.01\|x\|_2\}$.

Explain how a solution to the above problem alleviates in some sense, the problem encountered in question 3a, preferably using a picture.

(c) Show how to write the problem in such a way that the objective is linear.

(d) Show how to formulate the above problem in the form of a second-order cone program. (*Hint:* consider each constraint independently.)

4. *Proving convexity.* (4 pts) For given $v \in \mathbf{R}^n$, $S \in \mathbf{R}^{n \times p}$, we consider the following function

$$f(X) = 1 - v^T(I + SX^{-1}S^T)^{-1}v,$$

with $\mathbf{dom}\, f = \{X \in \mathbf{R}^{p \times p} \mid X = X^T \succ 0\}$. Show that f is convex.

Hint: Consider the epigraph of f and introduce a slack variable U such that $0 \prec U \preceq (I + SX^{-1}S^T)^{-1}$.