Midterm Solutions

1. *Dual of general LP.*

   (a) We first write the Lagrangian,

$$\begin{array}{rcl} L(x,\lambda,\nu) & = & c^T x + \lambda^T (Gx - h) + \nu^T (Ax - b) \\ & = & (c^T + \lambda^T G + \nu^T A)x - h\lambda^T - \nu^T b, \end{array}$$

   which is an affine function of x. The multiplier λ is positive componentwise (to ensure that the violation of the constraint $Gx \preceq h$ incurs an increase in the Lagrangian). The multiplier ν corresponding to the equality constraint has no sign constraint.

   (b) It follows that the dual function is given by

$$g(\lambda,\nu) = \inf_x L(x,\lambda,\nu) = \begin{cases} -\lambda^T h - \nu^T b & \text{if } c + G^T\lambda + A^T\nu = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

   (c) The Lagrange dual problem is, strictly speaking, given by

$$\begin{array}{ll} \text{maximize} & g(\lambda,\nu) \\ \text{subject to} & \lambda \succeq 0. \end{array}$$

   There is, however, one hidden equality constraint. The solution for the problem is $-\infty$ whenever $c + G^T\lambda + A^T\nu \neq 0$. We can, therefore, rewrite the problem as

$$\begin{array}{ll} \text{maximize} & -\lambda^T h - \nu^T b \\ \text{subject to} & c + G^T\lambda + A^T\nu = 0 \\ & \lambda \succeq 0 \end{array}$$

2. *A quadratic optimization problem.*

   (a) Define

$$L(x,\lambda) = x^T W x - \sum_{i=1}^n \lambda_i (x_i^2 - 1),$$

   where $\lambda_i \geq 0$. Note that, in this maximization problem, we want to have a decrease in the Lagrangian if the constraints are violated. With the choice of L above, we have

$$\inf_{\lambda \succeq 0} L(x,\lambda) = \begin{cases} x^T W x & \text{if } x_i^2 \leq 1, \quad i = 1,\dots,n, \\ -\infty & \text{otherwise.} \end{cases}$$

   This shows the desired result.

(b) This result follows from weak duality (note that the primal problem is not convex).

(c) The upper bound is

$$d^* = \inf_{\lambda \succeq 0} g(\lambda),$$

where g is the Lagrange dual function:

$$g(\lambda) = \sup_x L(x, \lambda).$$

For fixed λ, $L(x, \lambda)$ is quadratic in x. Therefore the above problem is an unconstrained quadratic optimization problem.
To solve it, we express $g(\lambda)$ as

$$g(\lambda) = \inf_{t,x} t \text{ subject to } t \geq L(x, \lambda) \text{ for every } x.$$

For given t, the constraint $t \geq L(x, \lambda)$ holds for every x if and only if

$$t \geq x^T W x - \sum_{i=1}^{n} \lambda_i (x_i^2 - 1) \text{ for every } x,$$

which is in turn equivalent to the linear matrix inequality in t, λ:

$$\begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \succeq \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} - \sum_{i=1}^{n} \lambda_i \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix},$$

or equivalently, with $S = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$:

$$\begin{bmatrix} S - W & 0 \\ 0 & t - \mathbf{Tr}\, S \end{bmatrix} \succeq 0.$$

We obtain that $g(\lambda)$ is the smallest t such that the above LMI in t, S holds with $S = \mathbf{diag}(\lambda) \succeq 0$. We obtain an equivalent form involving S only:

$$d^* = \min_S \mathbf{Tr}\, S \text{ subject to } S \text{ diagonal}, S \succeq W, \;\; S \succeq 0. \tag{1}$$

Note that the positivity constraint on S is redudant, since $W \succ 0$. This is consistent with the fact that the maximum value in the primal problem is attained at a vertex (since $x^T W x$ is convex in x). So, it does not make a difference if we replace inequalities by equalities in the original problem.

(d) This results from the fact that $\mathbf{Tr}\, WX = \mathbf{Tr}\, Wxx^T = x^T W x$.

(e) Relaxing the rank constraint, we get an upper bound for p^*:

$$\max \mathbf{Tr}\, WX \text{ subject to } X_{ii} = 1, \; 1 \leq i \leq n, \;\; X \succeq 0. \tag{2}$$

The dual of this SDP can be obtained by writing the Lagrangian:

$$L(X,S) = \mathbf{Tr}\, WX + \mathbf{Tr}\, QX - \mathbf{Tr}\, S(X-I),$$

where $Q \succeq 0$ accounts for the fact that X is constrained to be PSD, and S is a diagonal multiplier matrix that accounts for the equality constraints.

Maximizing the Lagrangian with respect to X for Q, S fixed yields $+\infty$, except when

$$0 = W + Q - S$$

In this case, the Lagrangian is $\mathbf{Tr}\, S$. The dual problem is therefore

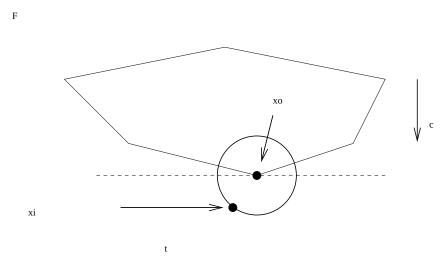
$$\min_S \ \mathbf{Tr}\, S \text{ subject to } S \text{ diagonal, } S - W = Q \succeq 0,$$

which is the desired result.

Note that there is no duality gap between problem (1) and (2) at the optimum, since at least one of the problems is strictly feasible. In fact, we can say more: since *both* problems are strictly feasible, both are attained, the duality gap is zero, and the KKT conditions are necessary and sufficient.

3. *LP with implementation constraints.*

   (a) What happens is that a small relative error translates in a large absolute error, if x is large. A picture explains this:



   (b) The modified problem makes sure that the constraints are still feasible despite the error we might make when implementing the solution. In addition, it seeks to minimize the effect of this error on the (worst-case) value of the objective.

   (c) The problem is equivalent to

$$\text{minimize } t \text{ subject to: for every } \delta \in \mathcal{B}(x), \quad \begin{array}{l} a_i^T(x+\delta) \le b_i \quad 1 \le i \le m \\ c^T(x+\delta) \le t \end{array}$$

   (d) For a vector a, and scalar b, we have

$$a^T(x+\delta) \le b \text{ for every } \delta \in \mathcal{B}(x)$$

if and only if

$$b - a^T x \geq \sup_{\|\delta\| \leq \rho \|x\|} a^T \delta = \rho \|a\| \cdot \|x\|,$$

where $\rho = 0.01$. The above is an SOCP constraint.
The robust LP is equivalent to the SOCP:

$$\text{minimize } t \text{ subject to } \begin{array}{l} a_i^T x + \rho \|a_i\| \|x\| \leq b_i \quad 1 \leq i \leq m \\ c^T x + \rho \|c\| \|x\| \leq t \end{array},$$

or, equivalently:

$$\text{minimize } c^T x + \rho \|c\| \|x\| \text{ subject to } a_i^T x + \rho \|a_i\| \|x\| \leq b_i, \ \ 1 \leq i \leq m.$$

4. *Proving convexity.* For $t \in \mathbf{R}$, $X \in \mathbf{dom}\, f$, we consider the inequality $f(X) \leq t$, or equivalently,

$$1 - t \leq v^T (I + S X^{-1} S^T)^{-1} v.$$

To see that the above constraint is jointly convex in X, t, note that the above holds if and only if there exist a matrix U such that

$$1 - t \leq v^T U v \text{ and } 0 \prec U \preceq (I + S X^{-1} S^T)^{-1}.$$

The second constraint is equivalent to

$$I + S X^{-1} S^T \preceq U^{-1}.$$

Multiplying on the left and right by $U \succ 0$ yields

$$U^2 + U S X^{-1} S^T U \preceq U.$$

Using Schur complements, we obtain an equivalent form:

$$\left[\begin{array}{ccc} U & US & U \\ S^T U & X & 0 \\ U & 0 & I \end{array} \right] \succeq 0$$

We conclude that f is convex, since its epigraph is the convex set

$$\mathbf{epi}\, f = \left\{ (X, t) \;\middle|\; \exists\, U \succ 0, \ \ \left[\begin{array}{ccc} U & US & U \\ S^T U & X & 0 \\ U & 0 & I \end{array} \right] \succeq 0, \ \ X \succ 0, \ \ 1 \leq t + v^T U v \right\}.$$