EE290N L. El Ghaoui 10/1/99

Homework Assignment #6

Due: Friday November 5, 1999 (before 2:00 pm please)

Reading assignment: chapter 5 of *Course Reader*.

1. *Project outline.* Describe what you intend to do in the project. Be sure to carefully describe what the problem is, what are the constraints, the variables. Don't concentrate on the *method* so much as on the *problem* you'd like to solve. You might first state a general (possibly hard) problem that you address, and then select one or more distinct problems that are (hopefully) easier to solve.

   The project may involve application of one of the standard problems seen in class (LP, SOCP, SDP, GP) in a specific application. Or, it may involve a specific algorithm (say, an interior-point method combined with quasi-Newton, or conjuguate gradients).

   - At this point, the project description is tentative, so don't think you are comitting yourself.
   - The final project should be fairly simple, don't try to solve your entire field's problems!

2. *Limits on concentration.* Consider the constraint on a decision variables $x \in \mathbf{R}^n_+$ that no more than 50% of its total can be concentrated in fewer than r entries. This can be expressed as

$$\sum_{i=1}^{r} x_{[i]} \leq 0.5 \sum_{i=1}^{n} x_i \tag{1}$$

   where $x_{[i]}$ is the ith largest component of x. (The fraction 50% is chosen arbitrarily; any other fraction could be used.)

   (a) Is the concentration constraint convex?

   (b) Show that the constraint (1) is equivalent to $\left(\begin{array}{c} n \\ r \end{array} \right)$ linear inequalities in x. What do you conclude about the complexity of a linear programming problem involving a concentration constraint?

   (c) Given a vector $x \in \mathbf{R}^m$, show that the sum of its r largest elements (*i.e.*, $x_{[1]} + \cdots + x_{[r]}$) is equal to the optimal value of the LP (in the variables $z \in \mathbf{R}^n$)

$$\begin{array}{ll} \text{maximize} & x^T z \\ \text{subject to} & 0 \preceq z \preceq \mathbf{1} \\ & \mathbf{1}^T z = r, \end{array} \tag{2}$$

   where $\mathbf{1} \in \mathbf{R}^n$ is the vector of ones.

(d) Derive the dual of the LP (2). Show that it can be written as

$$\begin{array}{ll} \text{minimize} & rt + \mathbf{1}^T y \\ \text{subject to} & t\mathbf{1} + y \succeq x \\ & y \succeq 0, \end{array} \tag{3}$$

where the variables are $t \in \mathbf{R}$, $y \in \mathbf{R}^m$.

(e) Show that the LP (3) has the same optimal value as (2), *i.e.*, $x_{[1]} + \cdots + x_{[r]}$.

(f) Using the previous results, show that the concentration constraint (1) can be represented as a set of $2n + 1$ linear inequalities in the $2n + 1$ variables x, y, t. What can you say now about the complexity of a linear programming problem involving a concentration constraint?