Lecture 10
Geometric programming

– geometric programming

– geometric programming in convex form

– applications

Geometric programming

monomial function:

$$f(x) = c x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

$c \geq 0$, $\alpha_j \in \mathbf{R}$, $\mathbf{dom}\, f = \{x \mid x \succ 0\}$

example: $f(x) = 2x_1^{2.1} x_2^{-1.7} x_3^{0.5}$

posynomial function:

$$f(x) = \sum_{k=1}^{r} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \cdots x_n^{\alpha_{nk}}$$

$c_k \geq 0$, $\alpha_{ik} \in \mathbf{R}$, $\mathbf{dom}\, f = \{x \mid x \succ 0\}$

example: $f(x_1, x_2, x_3) = x_1^2 x_2^{-1} x_3^{0.5} + 2x_1^{2.1} x_2^3$

geometric program

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, \ldots, m \\ & h_i(x) = 1, \quad i = 1, \ldots, p \end{array}$$

$f_0, \ldots, f_m$ posynomial, $h_1, \ldots, h_p$ monomial

not a convex problem

properties and examples

- posynomials closed under sums, products, nonnegative scaling
- monomials closed under sums, products, division, nonnegative scaling
- if f is a posynomial we say $1/f$ is inverse posynomial

examples:

- $0.1x_1x_3^{-1.2} + x_2^{0.5}x_3^{-0.8}$ is posynomial
- $1/(1 + x_2x_3^{-1.3})$ is inverse-posynomial
- $2x_3\sqrt{x1/x2}$ is monomial (hence both posy. and inverse-posy.)

GP in convex form

transformation of variables: $y_i = \log x_i$, $x_i = e^{y_i}$:

if $f(x) = cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ is monomial then

$$\log f(e^{y_1},\ldots,e^{y_n}) = a_1y_1 + \cdots + a_ny_n + b$$

(where $b = \log c$) is *affine* in y;

if $f(x) = \Sigma_{k=1}^r c_kx_1^{a_{1k}}x_2^{a_{2k}}\cdots x_n^{a_{nk}}$ is posynomial then

$$\log f(e^{y_1},\ldots,e^{y_n}) = \log \sum_{k=1}^r \exp(a_{1k}y_1 + \cdots + a_{nk}y_n + b_k)$$

(where $b_k = \log c_k$) is *convex* in y

geometric program in convex form

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{K_0} e^{a_{0k}^Ty + b_{0k}} \\ \text{subject to} & \sum_{k=1}^{K_i} e^{a_{ik}^Ty + b_{ik}} \le 1, \quad i = 1,\ldots,m \\ & e^{g_i^Ty + h_i} = 1, \quad i = 1,\ldots,p, \end{array}$$

where $a_{ik} \in \mathbf{R}^n$, $i = 0,\ldots,m$, contain the exponents in the posynomials of the original geometric program.

. . . a convex problem in standard form

Separable form

(assume no equality constraints $h_i(x) = 1$)

By introducing new variables, can obtain the

geometric program in separable form:

$$\begin{array}{ll} \text{minimize} & \Sigma_{i \in I_0} c_i \exp(x_i) \\ \text{subject to} & \Sigma_{i \in I_k} c_k \exp(x_k) \leq 1, \quad k = 1, \ldots, m, \\ & x = Ay \end{array}$$

Related problem

soon we'll prove that GP has a simple "dual" form

(again, assume no equality constraints $h_i(x) = 1$)

entropy maximization problem:

$$\begin{array}{ll} \text{maximize} & V(\delta) \\ \text{subject to} & A^T\delta = 0, \quad \Sigma_{i \in I_0} \delta_i = 1, \quad \delta \geq 0 \end{array}$$

where V is an entropy function

$$V(\delta) = \sum_{k=0}^{m} \sum_{i \in I_k} \delta_i \log \left(\sum_{i \in I_k} \delta_i \right) - \delta_i \log \delta_i$$

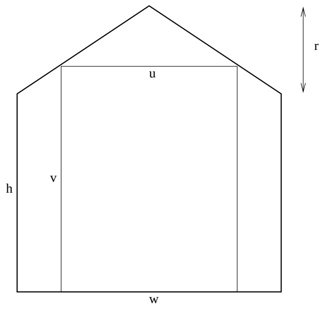
i.e., a **concave** function

In generic cases,

- the optimal values of the EM problem and of the GP problem are the same
- the variables of one are the Lagrange multipliers of the others

(see later ...)

Simple house design example



variables:

- width w
- wall height h
- roof height r

specs/constraints:

- enclose a given rectangle of width u and height u
- bounds on cost, proportional to lengths of floor & wall & roof
- upper & lower bounds on design variables w, h, r
- bounds on roof pitch $(2r/w)$

optimization problem

cost is of the form

$$\alpha w + 2\beta h + 2\gamma\sqrt{r^2 + (w/2)^2},$$

where

- α is the cost per unit length of the floor
- β is the cost per unit length of the walls
- γ is the cost per unit length of the roof

problem: maximize area of enclosed rectangle

$$\begin{array}{ll} \text{maximize} & uv \\ \text{subject to} & w_{\min} \le w \le w_{\max}, \\ & h_{\min} \le h \le h_{\max}, \\ & r_{\min} \le r \le r_{\max} \\ & p_{\min} \le 2r/w \le p_{\max} \\ & \alpha w + 2\beta h + 2\gamma\sqrt{r^2 + (w/2)^2} \le B \\ & h + ru/w \le v. \end{array}$$

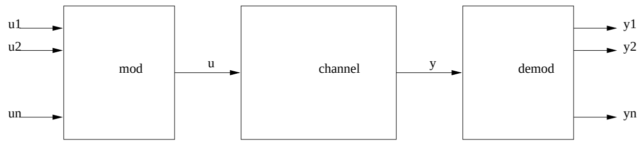
posynomial form

$$\begin{array}{ll}\text{maximize} & uv\\ \text{subject to} & w_{\min}/w \leq 1,\\ & h_{\min}/h \leq 1,\\ & r_{\min}/r \leq 1,\\ & p_{\min}w/(2r) \leq 1\\ & w/w_{\max} \leq 1,\\ & h/h_{\max} \leq 1,\\ & r/r_{\max} \leq 1,\\ & 2r/(wp_{\max}) \leq 1\\ & \alpha w/B + 2\beta h/B + 2\gamma t/B \leq 1\\ & h/v + ru/(wv) \leq 1\\ & r^2/t^2 + (w/2)^2/t^2 \leq 1.\end{array}$$

a geometric program (in posynomial form)

Power allocation in FDM system

frequency division multiplex (FDM) system



- signal u_i modulates carrier frequency f_i with power p_i
- channel is slightly nonlinear

demodulated signal power in y_i proportional to p_i

noise power in y_i is σ_i

interference power in y_i is sum of

- crosstalk
- intermodulation products from nonlinearity

problem: choose powers to maximize minimum SINR

crosstalk: crosstalk power c_i is linear in powers:

$$c = Cp, \quad C_{ij} \geq 0$$

C is usually tridiagonal, *i.e.*, crosstalk from adjacent channels only

intermodulation: kth order IM products have frequencies

$$\pm f_{i_1} \pm f_{i_2} \pm \cdots \pm f_{i_k}$$

with power proportional to $p_{i_1} p_{i_2} \cdots p_{i_k}$

e.g., for frequencies $1, 2, 3$:

frequency	IM product	pwr. prop. to
2	$1+1$	p_1^2
3	$1+2$	$p_1 p_2$
1	$2-1$	$p_2 p_1$
1	$3-2$	$p_3 p_2$
2	$3-1$	$p_3 p_1$
3	$1+1+1$	p_1^3
1	$1+1-1$	p_1^3
2	$2+1-1$	$p_2 p_1^2$
1	$2+2-3$	$p_2^2 p_3$
$\vdots$	$\vdots$	$\vdots$

total IM power at f_i is (complicated) polynomial of $p_1, \ldots, p_n$, with nonnegative coefficients

inverse SINR at frequency i

$$\frac{\text{noise} + \text{crosstalk} + \text{IM power}}{\text{signal power}}$$

is posynomial function of $p_1, \ldots, p_n$

hence, problem such as

$$\begin{array}{ll} \text{maximize} & \min_i \mathrm{SINR}_i \\ \text{subject to} & 0 < p_i \leq P_{\max} \end{array}$$

is geometric program

Transmitter power allocation (p. 5–14 ctd)

signal to interference/noise ratio (SINR), in dB:

$$\mathsf{SINR}_{ij} = 10 \log_{10} S_{ij}/I_{ij}$$

choose p_i to maximize average SINR in dB, *i.e.*:

$$\begin{array}{ll} \text{maximize} & \frac{1}{nm}\sum_{i,j} \mathsf{SINR}_{ij} \\ \text{subject to} & 0 \le p_i \le p_{\max} \end{array}$$

recast as geometric program:

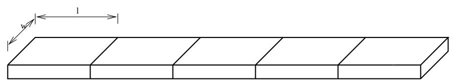
$$\begin{aligned} \frac{1}{nm}\sum_{i,j} \mathsf{SINR}_{ij} &= \frac{10}{nm} \log_{10} \prod_{i,j} (S_{ij}/I_{ij}) \\ &= \frac{10}{nm} \log_{10} \prod_{i,j} \frac{A_{iji}p_i}{\sum_{k \neq i} A_{ijk}p_k + N_{ij}} \end{aligned}$$

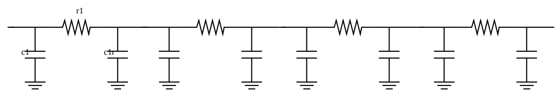
optimal p_j solve the geometric program

$$\begin{array}{ll} \text{minimize} & \prod_{i,j} \frac{\sum_{k\neq i} A_{ijk}p_k + N_{ij}}{A_{iji}p_i} \\ \text{subject to} & 0 \le p_i/p_{\max} \le 1 \end{array}$$

VLSI interconnect sizing

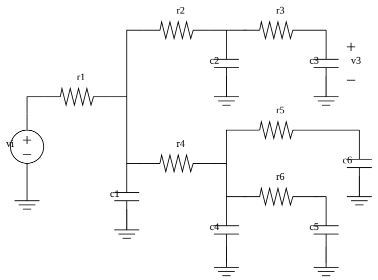
lumped RC model of interconnect wire:

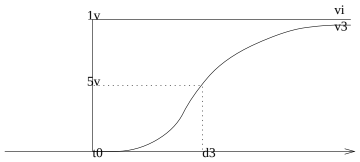




replace each segment with π model:
capacitance proportional to width, resistance inversely proportional to width

interconnect network: RC tree



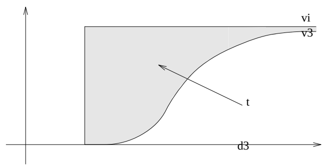


delay at node k: $D_k = \textbf{inf}\{T \mid v_k(t) \ge 0.5 \;\text{ for } t \ge T\}$

Elmore delay for RC tree

Elmore approximation of delay: area above step response

$$T_k = \int_0^\infty (1 - v_k(t))dt$$



in RC tree: Elmore delay to node k is

$$\sum_i C_i \left(\sum R\text{'s upstream from node } k \text{ and node } i\right)$$

example:

$$\begin{aligned} T_3 \;=\; & C_3(R_1 + R_2 + R_3) + C_2(R_1 + R_2) + C_1 R_1 \\ & + C_4 R_1 + C_5 R_1 + C_6 R_1 \end{aligned}$$

Elmore delay optimization via GP

in wire sizing

$$R_i = \alpha_i / w_i, \quad C_j = a_j^T w + b_j$$

$(\alpha_i \geq 0,\ a_j, b_j \geq 0)$

Elmore delay:

$$T_k = \sum_{ij} \gamma_{ij} R_j C_i = \sum_{k=1} \beta_k \prod_{i=1}^{m} w_i^{\alpha_{ik}}$$

$(\gamma_{ij} = +1$ or $0,\ \ \beta_k \geq 0,\ \alpha_{ij} = +1, 0, -1)$

...a *posynomial* function of $w \succ 0$

hence can minimize wire area subject to bound on Elmore delay using geometric programming:

$$\begin{array}{ll} \text{minimize} & \sum_i \ell_i w_i \\ \text{subject to} & T_k \leq T_{\max} \\ & w_{\min} \leq w_i \leq w_{\max} \end{array}$$