Chapter V
Some Examples

- geometrical problems
- data fitting and estimation
- problems in VLSI design
- filter design

Lecture 19
Geometrical problems

- projection & distance problems
- intersection, containment of polyhedra
- extremal volume ellipsoids
- classification problems
- S-procedure

Projection on convex set

projection of point x on set C defined as

$$P_C(x) = \mathrm{argmin}_{z \in C} \|x - z\|$$

i.e., point in C closest to x

suppose C has form

$$C = \{ x \mid Ax = b, \; f_i(x) \leq 0, \; i = 1, \ldots, m \}$$

$f_i : \mathbf{R}^n \to \mathbf{R}$ convex

$$\begin{array}{rll} P_C(x) = & \mathrm{argmin} & \|x - z\| \\ & \text{subject to} & Az = b \\ & & f_i(z) \leq 0, \; i = 1, \ldots, m \end{array}$$

computing $P_C(x)$ is cvx opt problem

Distance between convex sets

distance between sets C, $\tilde{C}$ defined as

$$\mathbf{dist}(C, \tilde{C}) = \min_{z \in C,\ \tilde{z} \in \tilde{C}} \|z - \tilde{z}\|$$

suppose C, $\tilde{C}$ are convex, with form

$$C = \{\ x \mid Ax = b,\ f_i(x) \le 0,\ i = 1, \ldots, m\ \}$$

$$\tilde{C} = \{\ x \mid \tilde{A}x = \tilde{b},\ \tilde{f}_i(x) \le 0,\ i = 1, \ldots, \tilde{m}\ \}$$

$f_i, \tilde{f}_i : \mathbf{R}^n \to \mathbf{R}$ convex

$\mathbf{dist}(C, \tilde{C})$ is optimal value of cvx problem

$$\begin{array}{ll} \text{minimize} & \|z - \tilde{z}\| \\ \text{subject to} & Az = b,\ \tilde{A}\tilde{z} = \tilde{b} \\ & f_i(z) \le 0,\ i = 1, \ldots, m \\ & \tilde{f}_i(\tilde{z}) \le 0,\ i = 1, \ldots, \tilde{m} \end{array}$$

Intersection & containment of polyhedra

inequality description

$$\begin{aligned} \mathcal{P}_1 &= \{x \mid a_i^T x \le b_i, i = 1, \ldots, m\} = \{x \mid Ax \preceq b\} \\ \mathcal{P}_2 &= \{x \mid f_i^T x \le g_i, i = 1, \ldots, l\} = \{x \mid Fx \preceq g\} \end{aligned}$$

- $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$? solve feasibility problem

$$Ax \preceq b, \quad Fx \preceq g$$

- $\mathcal{P}_1 \subseteq \mathcal{P}_2$? for $k = 1, \ldots, l$, check

$$\mathbf{sup}\{f_k^T x \mid Ax \preceq b\ \} \le g_k,$$

  i.e., solve LPs

$$\begin{array}{ll} \text{maximize} & f_k^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

vertex description

$$\mathcal{P}_1 = \textbf{Co}\,\{v_1, \ldots, v_K\}, \quad \mathcal{P}_2 = \textbf{Co}\,\{w_1, \ldots, w_L\}$$

– $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$? find $\lambda_1, \ldots, \lambda_K, \mu_1, \ldots, \mu_L \geq 0$ s.t.

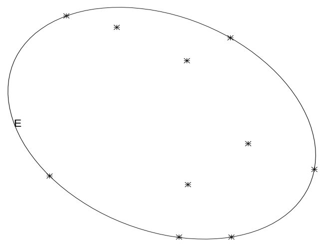
$$\begin{aligned} \lambda_1 + \cdots + \lambda_K &= 1 \\ \mu_1 + \cdots + \mu_L &= 1 \\ \lambda_1 v_1 + \cdots + \lambda_K v_k &= \mu_1 w_1 + \cdots + \mu_L w_l \end{aligned}$$

– $\mathcal{P}_1 \subseteq P_2$? for $k = 1, \ldots, K$ find $\mu_i \geq 0$, s.t.

$$\begin{aligned} v_k &= \mu_1 w_1 + \cdots + \mu_L w_L \\ 1 &= \mu_1 + \cdots + \mu_L \end{aligned}$$

Minimum volume ellipsoid around points

find minimum volume ellipsoid containing points $v_1, \ldots, v_K \in \mathbf{R}^n$



ellipsoid $\mathcal{E} = \{x \mid \|Ax - b\| \leq 1\}$

– center $A^{-1}b$
– $A = A^T \succ 0$, volume proportional to $\det A^{-1}$

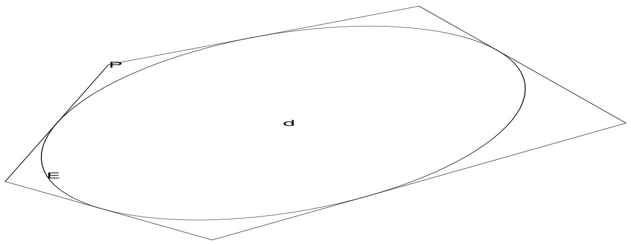
$$\begin{array}{ll} \text{minimize} & \log\det A^{-1} \\ \text{subject to} & A = A^T \succ 0 \\ & \|Av_i - b\| \leq 1, \quad i = 1, \ldots, K \end{array}$$

convex optimization problem in A, b
($n + n(n+1)/2$ vars)

Maximum volume ellipsoid in polyhedron

find maximum volume ellipsoid in

$$\mathcal{P} = \{x \mid a_i^T x \leq b_i,\ i = 1, \ldots, L\}$$



ellipsoid $\mathcal{E} = \{By + d \mid \|y\| \leq 1\}$

- center d
- $B = B^T \succ 0$, volume proportional to $\det B$

$$\begin{aligned} \mathcal{E} \subseteq \mathcal{P} &\iff a_i^T(By + d) \leq b_i \text{ for all } \|y\| \leq 1 \\ &\iff \sup_{\|y\| \leq 1} (a_i^T By + a_i^T d) \leq b_i \\ &\iff \|Ba_i\| + a_i^T d \leq b_i, \quad i = 1, \ldots, L \end{aligned}$$

convex constraint in B and d

maximum volume $\mathcal{E} \subseteq \mathcal{P}$, as convex problem in variables B, d:

$$\begin{array}{ll} \text{maximize} & \log\det B \\ \text{subject to} & B = B^T \succ 0 \\ & \|Ba_i\| + a_i^T d \leq b_i,\ i = 1, \ldots, L \end{array}$$

Löwner-John ellipsoid

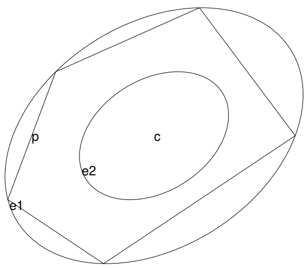
$C \subseteq \mathbf{R}^n$ convex, bounded

minimum volume ellipsoid containing C is called **Löwner-John ellipsoid** $\mathcal{E}_{\rm lj}$

fact. $\mathcal{E}_{\rm lj}$, shrunk by a factor n, lies inside C:

$$\frac{1}{n}(\mathcal{E}_{\rm lj} - x_{\rm lj}) + x_{\rm lj} \subseteq C \subseteq \mathcal{E}_{\rm lj}$$

where $x_{\rm lj} = \text{center}(\mathcal{E}_{\rm lj})$; if C is symmetric about any point, can replace n by $\sqrt{n}$



– arbitrarily complex convex sets can be approximated by ellipsoids
– ellipsoidal approximation 'universal' and 'efficient'
– worst-case bound depends only on dimension n
– factor $1/n$ tight for C a simplex

approximation by quadratic norm:
LJ result $\Longrightarrow$ *any* norm on $\mathbf{R}^n$ can be approximated within $\sqrt{n}$ by a quadratic norm, *i.e.*, there is $A \in \mathbf{R}^{n\times n}$ s.t.

$$\|Az\| \leq \|z\|_g \leq \sqrt{n}\|Az\|$$

efficiency of max-volume inscribed ellipsoid:

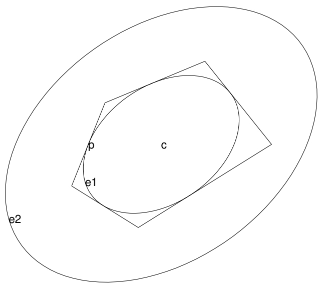
$C \subseteq \mathbf{R}^n$ convex, bounded, nonempty interior

$\mathcal{E}_{\rm mv}$ = max-volume ellipsoid inside C

fact. $\mathcal{E}_{\rm mv}$, expanded by factor n, covers C:

$$\mathcal{E}_{\rm mv} \subseteq C \subseteq n(\mathcal{E}_{\rm mv} - x_{\rm mv}) + x_{\rm mv}$$

where $x_{\rm mv}$ = center$(\mathcal{E}_{\rm mv})$



- another 'universal, efficient' ellipsoidal bound
- factor $1/n$ tight for C a simplex

Difficult variations

minor variations on these (convex, hence easy) problems are very difficult:

- maximum volume ellipsoid inside polyhedron $\mathbf{Co}\,\{v_1, \ldots, v_K\}$
- minimum volume ellipsoid containing polyhedron $Ax \preceq b$

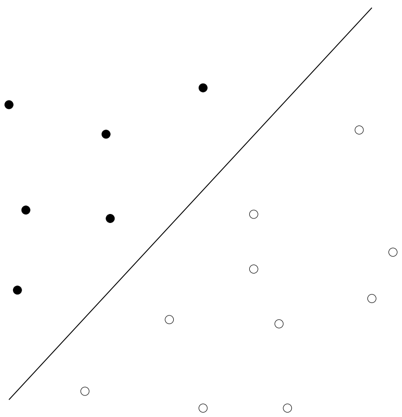
in fact, just checking whether a given ellipsoid $\mathcal{E}$ covers a polytope described by $\{x \mid Ax \preceq b\}$ is difficult (equivalent to *maximizing* convex quadratic s.t. linear inequalities)

Linear classification

separate two sets of points x^i and y^j by a hyperplane:

$$\begin{array}{ll} a^T x^i > t & \text{for all } i \\ a^T y^j < t & \text{for all } j \end{array}$$

LP feasibility problem in variables a, t

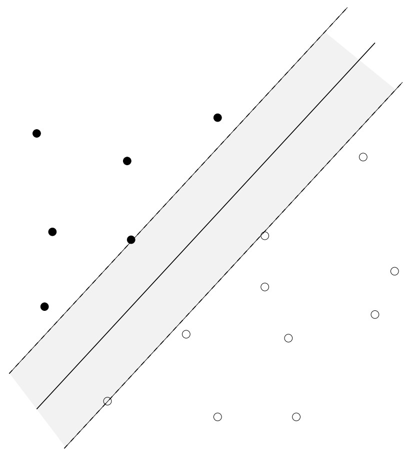


robust separation: separate by a slab of maximum thickness

$$\begin{array}{ll} \text{maximize} & \min_i a^T x_i - \max_j a^T y_j \\ \text{subject to} & \|a\| \leq 1 \end{array}$$

or

$$\begin{array}{ll} \text{maximize} & t_1 - t_2 \\ \text{subject to} & a^T x_i \geq t_1 \text{ for all } i \\ & a^T y_j \leq t_2 \text{ for all } j \\ & \|a\| \leq 1 \end{array}$$



Nonlinear classification

separate two sets of points by a quadratic surface:

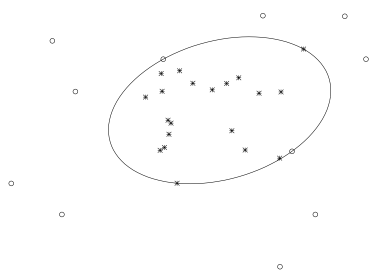
$$(x^i)^T A x^i + b^T x^i + c < 0 \text{ for all } i$$
$$(y^j)^T A y^j + b^T y^j + c > 0 \text{ for all } j$$

(a set of linear inequalities in variables $A = A^T$, b, c)

extra constraints:

- *ellipsoid* containing x's, not y's: add constraint $A \succ 0$
- optimize size or shape

example find most spherical ellipsoid containing x's, not y's



example, ctd.

- condition number $\kappa = \lambda_{\max}(A)/\lambda_{\min}(A)$ measures eccentricity of ellipsoid
- can normalize $A \succeq I$ ($\lambda_{\min}(A) \geq 1$)
- objective is $\lambda_{\max}(A)$

SDP in $A = A^T$, b, c, and γ:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & (x^i)^T A x^i + b^T x^i + c \leq 0 \text{ for all } i \\ & (y^j)^T A y^j + b^T y^j + c \geq 0 \text{ for all } j \\ & I \preceq A \preceq \gamma I \end{array}$$

A useful result for quadratic functions

$$S_i = \{x \mid x^T A_i x + 2b_i^T x + c_i \le 0\}, \; i = 0, 1$$

(no assumption of convexity)

problem: when do we have $S_1 \subseteq S_0$?

sufficient condition: there exists $\tau \ge 0$

$$\begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 \end{bmatrix} \preceq \tau \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix}$$

(easy) **proof.** condition means for all x,

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \le \tau \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

i.e., for all x,

$$x^T A_0 x + 2b_0^T x + c_0 \le \tau (x^T A_1 x + 2b_1^T x + c_1)$$

therefore,

$$x^T A_1 x + 2b_1^T x + c_1 \le 0 \implies x^T A_0 x + 2b_0^T x + c_0 \le 0$$

condition is also necessary if S_1 has nonempty interior, *i.e.*, there is a point x with

$$x^T A_1 x + 2b_1^T x + c_1 < 0$$

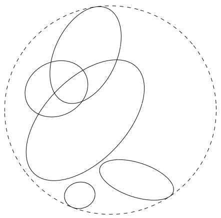
(proof is not easy)

called S-procedure (in control theory)

Example

find smallest ball $\mathcal{B}$ that contains given ellipsoids $\mathcal{E}_i$

$\mathcal{B}$: $x^T x - 2x_c^T x + \gamma \le 0$
radius $(\mathcal{B}) = \sqrt{x_c^T x_c - \gamma}$
$\mathcal{E}_i$: $x^T A_i x + 2b_i^T x + c_i \le 0$



$\mathcal{B}$ contains $\mathcal{E}_i$ iff there exists $\tau_i \ge 0$:

$$\begin{bmatrix} I & -x_c \\ -x_c^T & \gamma \end{bmatrix} \preceq \tau_i \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix}$$

convex problem in x_c, γ, τ_i:

$$\begin{array}{ll} \text{minimize} & x_c^T x_c - \gamma \\ \text{subject to} & \begin{bmatrix} I & -x_c \\ -x_c^T & \gamma \end{bmatrix} \preceq \tau_i \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \\ & \tau_i \ge 0 \end{array}$$

S-procedure for many quadratic functions

$$S_i = \{x \mid x^T A_i x + 2b_i^T x + c_i \le 0\}, \; i = 0, \ldots, L$$

(no assumption of convexity)

problem: $S_1 \cap \cdots \cap S_L \subseteq S_0$?

sufficient condition: there exist $\tau_1, \ldots, \tau_L \ge 0$ s.t.

$$\begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 \end{bmatrix} \preceq \tau_1 \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} + \cdots + \tau_L \begin{bmatrix} A_L & b_L \\ b_L^T & c_L \end{bmatrix}$$

proof. condition implies for all x,

$$x^T A_0 x + 2b_0^T x + c_0 \le \sum_{i=1}^{L} \tau_i (x^T A_i x + 2b_i^T x + c_i)$$

Hence, $x^T A_i x + 2b_i^T x + c_i \le 0$, $i = 1, \ldots, L$ implies $x^T A_0 x + 2b_0^T x + c_0 \le 0$

- condition not necessary (except when $L = 1$)
- still, leads to useful convex relaxations

Application in combinatorial optimization

general quadratically constrained quadratic optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \ i = 1, \ldots, L \end{array}$$

$f_i(x) = x^T A_i x + 2b_i^T x + c_i$

- very hard problem if A_i's are not positive semidefinite
- can compute lower bound via convex optimization

t is lower bound if

$$f_i(x) \le 0, \quad i = 1, \ldots, L \implies f_0(x) \ge t$$

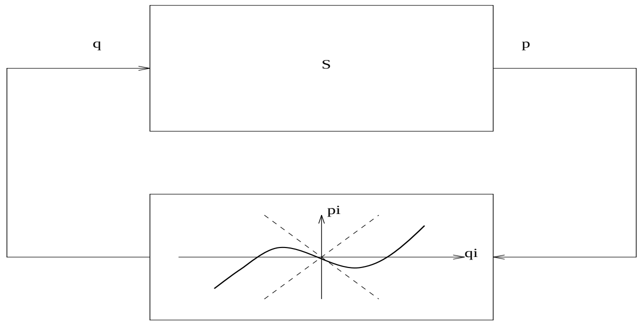
sufficient condition: exist $\tau_i \ge 0$, $i = 1, \ldots, L$, s.t.

$$\begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 - t \end{bmatrix} + \sum_{i=1}^{L} \tau_i \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \succeq 0$$

linear matrix inequality in τ_i

finding best lower bound is a semidefinite program

Application in control



nonlinearities satisfy: $u_i^2(t) \le y_i^2(t)$, $i = 1, \ldots, M$

$V(t) = x(t)^T P x(t)$ ($P \succ 0$) is a Lyapunov function if

$$dV/dt = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} < 0$$

for all nonzero (x, u) s.t. $u_i^2 \le (Cx)_i^2$, $i = 1, \ldots, M$

sufficient condition via $\mathcal{S}$-procedure

$$\begin{bmatrix} A^T P + PA + C^T DC & PB \\ B^T P & -D \end{bmatrix} \succ 0$$

LMI in $P = P^T$, diagonal D