Lecture 26

Robust semidefinite programming: part two

Principle

– consider a robust SDP, with a data uncertainty set that satisfies LFR assumption and property (P)

(*e.g.*, assume data depends algebraically on unknown-but-bounded parameters, with Euclidean-norm bounds)

– using LFRs, represent robust SDP as a quadratically constrained program (or rank-constrained SDP)

– using Lagrange (or rank) relaxation, get SDP approximation to original problem

– this SDP yields a bound (and a robust solution) to the original problem

Robust SDP: LFR framework

nominal SDP:

$$\min\ c^T x \text{ subject to } F(x) \succeq 0$$

where symmetric $F(x)$ is affine in x

robust SDP:

$$\min c^T x \quad : \ \mathbf{F}(x, \Delta) \succeq 0 \text{ for every } \Delta \in \mathbf{\Delta}$$

where $\mathbf{F}(x, \Delta)$ has LFR

$$\mathbf{F}(x, \Delta) = F(x) + \ L(x)\Delta(I - D\Delta)^{-1}R + \left(L(x)\Delta(I - D\Delta)^{-1}R\right)^T$$

- symmetric $F(x)$ and $L(x)$ are affine in x (ensures convexity of robust SDP)
- R, $D \in \mathbf{R}^{n_q \times n_p}$ fixed matrices
- "uncertainty set" $\mathbf{\Delta}$ is a given subset of $\mathbf{R}^{n_p \times n_q}$
- $\mathbf{\Delta}$ satisfies property (P)
- **technical assumption:** LFR is well-posed over $\mathbf{\Delta}$

Robustness analysis

fix x, and seek sufficient condition for

$$\xi^T \mathbf{F}(x, \Delta)\xi \geq 0 \text{ for every } \xi, \ \ \Delta \in \mathbf{\Delta} \qquad (*)$$

define

$$p = \Delta q, \text{ where } q = (I - D\Delta)^{-1} R\xi$$

with well-posedness asumption, equivalent to

$$p = \Delta q, \ \ q = R\xi + Dp$$

thus, condition $(*)$ is equivalent to

$$\xi^T \left(F(x)\xi + 2L(x)p\right) \geq 0$$

for every ξ, p, q, Δ such that

$$q = R\xi + Dp, \ \ p = \Delta q, \ \ \Delta \in \mathbf{\Delta}$$

Equivalent form

Since $\Delta \in \mathbf{\Delta}$, and $\mathbf{\Delta}$ satisfies property (P), we have

(p, ξ) satisfy $p = \Delta(R\xi + Dp)$ for some $\Delta \in \mathbf{\Delta}$

if and only if

$$\Phi_{\mathbf{\Delta}}\left(\begin{bmatrix} R & D \\ 0 & I \end{bmatrix}\begin{bmatrix} \xi \\ p \end{bmatrix}\begin{bmatrix} \xi \\ p \end{bmatrix}^T\begin{bmatrix} R & D \\ 0 & I \end{bmatrix}\right) \succeq 0$$

for fixed x, necessary and sufficient condition for

$$\xi^T \mathbf{F}(x, \Delta)\xi \geq 0 \text{ for every } \xi, \quad \Delta \in \mathbf{\Delta} \qquad (*)$$

is thus:

$$\xi^T (F(x)\xi + 2L(x)p) \geq 0 \text{ for every } \xi, p$$

$$\text{such that } \Phi_{\mathbf{\Delta}}\left(\begin{bmatrix} R & D \\ 0 & I \end{bmatrix}\begin{bmatrix} \xi \\ p \end{bmatrix}\begin{bmatrix} \xi \\ p \end{bmatrix}^T\begin{bmatrix} R & D \\ 0 & I \end{bmatrix}\right) \succeq 0.$$

note that

$$\xi^T (F(x)\xi + 2L(x)p) = \begin{bmatrix} \xi \\ p \end{bmatrix}^T \begin{bmatrix} F(x) & L(x) \\ L(x)^T & 0 \end{bmatrix}\begin{bmatrix} \xi \\ p \end{bmatrix}$$

Result

We have obtained the following result:

$$\xi^T \mathbf{F}(x, \Delta)\xi \geq 0 \text{ for every } \xi, \quad \Delta \in \mathbf{\Delta}$$

if and only if

$$\begin{bmatrix} \xi \\ p \end{bmatrix}^T \begin{bmatrix} F(x) & L(x) \\ L(x)^T & 0 \end{bmatrix}\begin{bmatrix} \xi \\ p \end{bmatrix} \geq 0$$

for every ξ, p such that

$$\Phi_{\mathbf{\Delta}}\left(\begin{bmatrix} R & D \\ 0 & I \end{bmatrix}\begin{bmatrix} \xi \\ p \end{bmatrix}\begin{bmatrix} \xi \\ p \end{bmatrix}^T\begin{bmatrix} R & D \\ 0 & I \end{bmatrix}\right) \succeq 0$$

x will be robustly feasible if and only if a (scalar) quadratic inequality holds whenever some other quadratic (matrix) inequality holds

from there, two (generically equivalent) ways:

- Lagrange relaxation
- rank relaxation

Lagrange relaxation

previous condition holds if there exist a positive semidefinite matrix S such that :

for every ξ, p

$$\begin{bmatrix} \xi \\ p \end{bmatrix}^T \begin{bmatrix} F(x) & L(x) \\ L(x)^T & 0 \end{bmatrix} \begin{bmatrix} \xi \\ p \end{bmatrix} \geq \mathbf{Tr}\, S\Phi_{\mathbf{\Delta}} \left(\begin{bmatrix} R & D \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ p \end{bmatrix} \begin{bmatrix} \xi \\ p \end{bmatrix}^T \begin{bmatrix} R & D \\ 0 & I \end{bmatrix}^T \right)$$

Since RHS is quadratic in ξ, p:

$$\mathbf{Tr}\, S\Phi_{\mathbf{\Delta}} \left(\begin{bmatrix} R & D \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ p \end{bmatrix} \begin{bmatrix} \xi \\ p \end{bmatrix}^T \begin{bmatrix} R & D \\ 0 & I \end{bmatrix}^T \right) = \begin{bmatrix} \xi \\ p \end{bmatrix}^T \begin{bmatrix} R & D \\ 0 & I \end{bmatrix}^T \Phi_{\mathbf{\Delta}}^*(S) \begin{bmatrix} R & D \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ p \end{bmatrix},$$

(here $\Phi_{\mathbf{\Delta}}^*$ is the dual of $\Phi_{\mathbf{\Delta}}$)

condition above involves a single, scalar quadratic inequality in ξ, p:

for every ξ, p

$$\begin{bmatrix} \xi \\ p \end{bmatrix}^T \left(\begin{bmatrix} F(x) & L(x) \\ L(x)^T & 0 \end{bmatrix} - \begin{bmatrix} R & D \\ 0 & I \end{bmatrix}^T \Phi_{\mathbf{\Delta}}^*(S) \begin{bmatrix} R & D \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} \xi \\ p \end{bmatrix} \geq 0$$

SDP approximation

or, equivalently,

$$\begin{bmatrix} F(x) & L(x) \\ L(x)^T & 0 \end{bmatrix} \succeq \begin{bmatrix} R & D \\ 0 & I \end{bmatrix}^T \Phi_{\mathbf{\Delta}}^*(S) \begin{bmatrix} R & D \\ 0 & I \end{bmatrix} \qquad (1)$$

hence, Lagrange relaxation of robust SDP:

$$\min c^T x \;\; \text{subject to } S \succeq 0 \text{ and } (1)$$

above is an SDP in variables x, S, which provides an approximation of the robust SDP:

- yields a lower bound on the robust SDP
- optimal x is robust (feasible for all $\Delta \in \mathbf{\Delta}$)
- projection of feasible set of above problem into space of x-variables is an *inner* approximation of robust feasible set

when $\Phi(\cdot)$ takes scalar values, approximation is exact

Recovering earlier results

In a previous lecture, we considered a special case:

$$\mathbf{\Delta} = \{\Delta = \mathbf{diag}\,(\delta_1, \ldots, \delta_l) \;\mid\; \|\Delta\| \le 1\}$$

and $\mathbf{F}(x, \Delta)$ is affine in Δ: (hence, $D = 0$)

$$\mathbf{F}(x, \Delta) = F(x) + L(x)\Delta R(x) + (L(x)\Delta R(x))^T$$

where $F(\cdot)$, $L(\cdot)$ and $R(\cdot)$ affine in x (note we include symmetric part)

The robust SDP approximation was

$$\min_{S,x} c^T x \quad : \; S \text{ diagonal}, \quad \begin{bmatrix} F(x) - R^T S R & L(x) \\ L(x)^T & S \end{bmatrix} \succeq 0$$

Is it the same?

Recovering previous results

In the previous case, the mapping $\Phi_{\mathbf{\Delta}}$

$$\Phi_{\mathbf{\Delta}}\left(\begin{bmatrix} q \\ p \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix}^T\right) = \mathbf{diag}\,(q_i^2 - p_i^2)_{i=1}^l \succeq 0$$

(Indeed, $p = \Delta q$ for some diagonal Δ, $\|\Delta\| \le 1$, iff $q_i^2 \ge p_i^2$)

$\Phi_{\mathbf{\Delta}}$ transforms $2l \times 2l$ symmetric matrices to $l \times l$ diagonal matrices

The dual of $\Phi_{\mathbf{\Delta}}$ is takes a $l \times l$ diaongal matrix as input

$$\Phi_{\mathbf{\Delta}}^*(S) = \begin{bmatrix} S & 0 \\ 0 & -S \end{bmatrix}$$

(to prove this, check that

$$\mathbf{Tr}\, S\Phi_{\mathbf{\Delta}}(X) = \mathbf{Tr}\, \Phi_{\mathbf{\Delta}}^*(S) X$$

holds for every S diagonal and X of appropriate size.)

Specializing the Lagrange relaxation result with $D = 0$, and the above dual mapping, results in previous result

Rank relaxation

define

$$N := \begin{bmatrix} R & D \\ 0 & I \end{bmatrix},$$

$$M(x) := \begin{bmatrix} F(x) & L(x) \\ L(x)^T & 0 \end{bmatrix} \left(= M_0 + \sum_{i=1}^m y_i M_i \right)$$

$$\mathcal{X} := \left\{ X \;\middle|\; \Phi_{\mathbf{\Delta}}\left(NXN^T\right) \succeq 0, \quad X \succeq 0 \right\}$$

(note that $\mathcal{X}$ contains 0 since $\Phi_{\mathbf{\Delta}}$ is linear)

robust SDP is

$$p^* := \min \; c^T x \text{ subject to}$$
$$\mathbf{Tr}\, M(x)X \geq 0 \text{ whenever } \begin{cases} X \in \mathcal{X}, \\ \mathbf{Rank} X = 1 \end{cases}$$

Rank relaxation: drop the rank constraint to obtain an upper bound on objective

$$p^* \leq \min \; c^T x \text{ subject to}$$
$$\mathbf{Tr}\, M(x)X \geq 0 \text{ whenever } X \in \mathcal{X}$$

Exchanging min and max

Above problem can be written in min-max form as

$$p^* \leq \min_x \max_{X \in \mathcal{X}} \; c^T x - \mathbf{Tr}\, M(x)X$$

Since objective function is saddle function, we can exchange min and max:

$$p^* \leq \max_{X \in \mathcal{X}} \; \min \; c^T x - \mathbf{Tr}\, M(x)X$$

Can express the "min" part analytically: for every $X \in \mathcal{X}$, we have

$$\min_x \; c^T x - \mathbf{Tr}\, M(x)X = \begin{cases} -\mathbf{Tr}\, M(0)X & \text{if } \mathbf{Tr}\, M_i X = c_i, \\ & i = 1, \ldots, m \\ -\infty & \text{else} \end{cases}$$

Hence, upper bound computation problem is

$$\max -\mathbf{Tr}\, M(0)X \quad : \quad \begin{array}{l} \mathbf{Tr}\, M_i X = c_i, \; 1 \leq i \leq m, \\ \Phi_{\mathbf{\Delta}}\left(NXN^T\right) \succeq 0, \quad X \succeq 0 \end{array}$$

which is the dual of the SDP obtained via Lagrange relaxation

Quality of approximation

quality of approximation depends on uncertainty structure and bounds

- in the unstructured case, approximation is exact
- results exist when the uncertainty enters *affinely* in the data

Theorem (Nemirovski, 1997)

let $F_0 \succeq 0,\ F_1, \ldots, F_l$ be given symmetric matrices, we have

$$\mathbf{F}(\delta) := F_0 + \sum_{i=1}^{l} \delta_i F_i \succeq 0 \text{ for every } \delta, \quad \delta^T \delta \leq 1$$

if there exist $S \succ 0$ such that

$$2F_0 \succeq S + \sum_{i=1}^{l} F_i S^{-1} F_i \qquad (1)$$

the above result is α-conservative, with $\alpha = \min(\sqrt{n}, \sqrt{l})$, in the sense that

$$\mathbf{F}(\delta) := F_0 + \sum_{i=1}^{l} \delta_i F_i \succeq 0 \text{ for every } \delta, \quad \delta^T \delta \leq \alpha^2$$

implies (1).

Proof of conservativeness estimate

(WLOG, $F_0 = I$)

if (1) does not hold, $\exists\ h \in \mathbf{R}^n$ s. t.

$$1 = h^T h < \sum_{i=1}^{l} h^T F_i^2 h = \mathbf{Tr}\,(H^T H),$$

where $H = [F_1 h \ldots F_l h] \in \mathbf{R}^{n \times l}$

with $\alpha = \min(\sqrt{n}, \sqrt{l})$, we have

$$\|H\|_F \leq \alpha \|H\|,$$

which means that $\exists \delta \in \mathbf{R}^l$ such that

$$1 < \mathbf{Tr}\,(H^T H) \leq \|H\delta\|^2, \quad \delta^T \delta = \alpha^2.$$

since $H\delta = \left(\sum_{i=1}^{l} \delta_i F_i\right) h$, we obtain that $\exists \delta \in \mathbf{R}^l$ which satisfies

$$\left\| \sum_{i=1}^{l} \delta_i F_i \right\| > 1, \quad \delta^T \delta = \alpha^2,$$

which is the desired contradiction.

Link with Lyapunov theory in control

example: stability of

$$\dot{q} = Aq,$$

where $q \in \mathbf{R}^n$ is the state, and A is a (constant) square matrix

(asymptotic) stability means: **every** trajectory of the system goes to zero as $t \to \infty$

interpretation: whatever the initial condition $q(0)$, system goes back to equilibrium ($q = 0$)

classical result: system is stable iff every eigenvalue of A has negative real part

in other words, system is stable iff

$$(sI - A)^H (sI - A) \succ 0 \text{ for every } s, \quad s + s^* \geq 0.$$

(symbol ^{H} stands for transpose conjuguate)

Associated uncertainty set

introduce $p = sq$, we want to check if

$$\|sq - Aq\|^2 > 0 \text{ for every } q \neq 0, \quad s + s^* \geq 0$$

This is a **robustness analysis** problem with "uncertain parameter" s

In this case, the uncertainty set $\mathbf{\Delta}$ is

$$\mathbf{\Delta} = \{\Delta = s \cdot I_n \mid s + s^* \geq 0\},$$

and the corresponding map $\Phi_{\mathbf{\Delta}}$ is characterized by

$$\Phi_{\mathbf{\Delta}} \left(\begin{bmatrix} q \\ p \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}^H \right) = pq^H + qp^H$$

proof: for every vectors $p, q \in \mathbf{C}^n$, we have

$$p = sq \text{ for some } s, \quad s + s^* \geq 0 \Longleftrightarrow pq^H + qp^H \succeq 0$$

Lagrange relaxation

Our problem is thus to check if

$\|p - Aq\|^2 > 0$ for every $(p, q) \neq (0, 0)$, $\;pq^H + qp^H \succeq 0$

via Lagrange relaxation, get a sufficient condition for stability: $\exists\ S \succ 0$ such that

$$\forall (p, q) \neq (0, 0), \quad \|p - Aq\|^2 > \textbf{Tr}\, S \left(pq^H + qp^H \right)$$

try to show: above equivalent to classical Lyapunov condition:

$$\exists\ S \text{ such that } A^T S + SA \prec 0, \quad S \succ 0$$

Dynamical interpretation

The quadratic function

$$V(x) = x^T S x$$

has a nice interpretation: along every non-zero trajectory, V decreases:

$$\frac{d}{dt} V(x(t)) = x(t)^T \left(A^T S + SA \right) x(t) < 0,$$

in other words: V acts as a 'total energy" that proves stability

can prove that condition is also necessary

An extension: linear system with delay

$$\dot{x} = Ax(t) + A_d x(t-\tau)$$

where $\tau \geq 0$ is the delay, is stable for every $\tau \geq 0$ iff

$$\det(sI - A - zA_d) \neq 0$$

for every s, z, $\Re(s) \geq 0$, $|z| \leq 1$

use same approach with 2 parameters

our problem is thus to check if

$$\|p - Aq - A_d p_d\|^2 > 0 \text{ for every } (p, p_d, q) \neq (0, 0), \quad pq^H + qp^H \succeq 0, \ \ p_d p_d^H \preceq qq^H$$

apply Lagrange relaxation from there ...

this time, condition is only sufficient

Robust quadratic constraints

Sometimes we'll encounter constraints that are of the form

$$\mathbf{A}^T W \mathbf{A} \preceq 0 \text{ for every } \mathbf{A} \in \mathcal{U}$$

where $\mathbf{A}$ is described by an LFR, and $W = W^T$ is a (nonnecessarily PSD) matrix

Above constraint is not convex in general (even with no uncertainty)

We can generalize the Lagrange relaxation technique to handle this case, provided (as usual)

- $\mathcal{U}$ described by well-posed LFR
$$\mathcal{U} = \left\{ \mathbf{A}(\Delta) = A + L\Delta(I - D\Delta R)^{-1} \ \mid \Delta \in \mathbf{\Delta} \right\},$$
- $\mathbf{\Delta}$ satisfies property (P)

Quadratic matrix inequality robustness lemma

For a given real, symmetric matrix of appropriate size W, the condition

$$\mathbf{A}^T W \mathbf{A} \preceq 0 \text{ for every } \mathbf{A} \in \mathcal{U}$$

holds if there exist a matrix S of appropriate size such that

$$\begin{array}{l} \left[\begin{array}{c} A^T \\ L^T \end{array}\right] W \left[\begin{array}{cc} A & L \end{array}\right] + \left[\begin{array}{cc} R & D \\ 0 & I \end{array}\right]^T \Phi^*_{\mathbf{\Delta}}(S) \left[\begin{array}{cc} R & D \\ 0 & I \end{array}\right] \preceq 0, \\ S \succeq 0 \end{array}$$

proof: Lagrange relaxation